12.5 MODELS OF CONVECTION IN LAMINAR FLOWS

In a number of practical and important situations, viscous fluids pass in laminar flow across surfaces with which they exchange heat. In this section we develop and examine some models for the convective heat transfer coefficient under laminar flow.

12.5.1 A Parallel Plate Heat Exchanger with Isothermal Surfaces

Figure 12.5.1 shows a laminar flow confined between parallel planes, which are maintained at a fixed temperature $T_H$. A fluid at a uniform inlet temperature $T_i$ enters the heated section and flows at a mean velocity $U$ down the channel. We neglect any effect of temperature variation on the viscosity of the fluid.

The flow is taken to be laminar and Newtonian, and it is easily shown that the velocity profile is

$$u(y) = \frac{3}{2} U \left[ 1 - \left( \frac{y}{H} \right)^2 \right] \quad (12.5.1)$$

where $U$ is the average velocity across the channel. Hence the volumetric flowrate per unit width of channel is

$$q_w = 2 U H \quad (12.5.2)$$

We assume, in this analysis, that the channel is very wide compared to $H$ and that the flow and the heat transport are two-dimensional (i.e., no variations in the $z$ direction).

The convective energy equation in the region $[-H, H]$ for steady state operation takes the form

$$u(y) \frac{\partial T}{\partial x} = \alpha \left( \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} \right) \quad (12.5.3)$$

The rate of heat conduction along the axis (the $\partial^2 T/\partial x^2$ term in Eq. 12.5.3) is generally much smaller than the rate at which heat is transferred by convection (the term on the

![Figure 12.5.1 Laminar flow between heated parallel planes.](image-url)
left-hand side). Hence a good approximation (see Problem 12.65) is to neglect axial conduction and write Eq. 12.5.3 in the form

$$u(y) \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (12.5.4)$$

As always, the specification of boundary conditions determines much of the character of the solution to this equation. Hence we discuss these conditions now.

We must supply an entrance condition regarding the temperature of the fluid at the plane $x = 0$. We take the temperature to be uniform there, and write

$$T = T_1 \quad \text{at} \quad x = 0 \quad \text{for all} \; y \quad (12.5.5)$$

We assume symmetry along the midplane of the channel, and so we write

$$\frac{\partial T}{\partial y} = 0 \quad \text{at} \quad y = 0 \quad \text{for all} \; x \quad (12.5.6)$$

At the fluid/solid interface we take the case of an isothermal wall, so we write

$$T = T_H \quad \text{at} \quad y = \pm H \quad \text{for all} \; x \geq 0 \quad (12.5.7)$$

We now have enough boundary conditions to permit a solution to the convective energy equation. Note that the equation (Eq. 12.5.4) is linear, but because of the velocity term it has nonconstant coefficients. The equation, with this set of linear boundary conditions, can be solved analytically by a series method. Mathematically, we call the equation and boundary conditions a Sturm–Liouville problem, for which a solution procedure can be found in a number of applied mathematics texts. We simply present the solution here, without derivation. (We treated a very similar problem in Section 5.3 of Mass Transfer.)

We first nondimensionalize the equations in the following manner. We define a dimensionless temperature as

$$\Theta = \frac{T - T_H}{T_1 - T_H} \quad (12.5.8)$$

and dimensionless space variables

$$y^* = \frac{y}{H} \quad x^* = \frac{x}{UH^2} \quad (12.5.9)$$

Now the differential equation to be solved takes the form

$$\frac{3}{2} (1 - y^{*2}) \frac{\partial \Theta}{\partial x^*} = \frac{\partial^2 \Theta}{\partial y^{*2}} \quad (12.5.10)$$

and the boundary conditions are simplified to

$$\Theta = 1 \quad \text{at} \quad x^* = 0 \quad \text{for all} \; y^* \quad (12.5.11)$$

$$\frac{\partial \Theta}{\partial y^*} = 0 \quad \text{at} \quad y^* = 0 \quad \text{for all} \; x^* \quad (12.5.12)$$

$$\Theta = 0 \quad \text{at} \quad y^* = \pm 1 \quad \text{for all} \; x^* \geq 0 \quad (12.5.13)$$

An analytical solution for $\Theta$ can be obtained, which takes the form of a product of an infinite series of polynomials in $y^*$ times exponentially decaying functions of $x^*$. The form of the solution is

$$\Theta = \sum_{m=0}^{\infty} \left[ A_m \exp \left( -\frac{2\lambda_m^2 x^*}{3} \right) \sum_{n=0}^{\infty} \sigma_{mn} y^{*n} \right] \quad (12.5.14)$$
We are not really interested in the \( y^* \) dependence of \( \Theta \). What we do want to know is the change in mean temperature of the fluid accomplished by passage of the fluid down a length \( L \) of the channel. A simple energy balance tells us that the rate of heat transfer, averaged over the length of the channel, is (per unit width of channel)

\[
Q_H = 2UH \rho C_p (T_1 - T_{cm})
\]

(12.5.15)

where \( T_{cm} \) is (except for notation) the cup-mixing average defined earlier (see Eq. 12.4.8). Hence we need to obtain the cup-mixing average from the temperature profile given in Eq. 12.5.14, so we must evaluate the integral (in terms of \( \Theta \) instead of \( T \))

\[
\Theta_{cm} = \frac{\int_0^H \frac{y}{H} U \left( 1 - \left[ \frac{y}{H} \right]^2 \right) \Theta(x, y) \, dy}{\int_0^H \frac{y}{H} U \left( 1 - \left[ \frac{y}{H} \right]^2 \right) \, dy}
\]

(12.5.16)

It is possible to show that this can be written in the form

\[
\Theta_{cm} = \sum_{m=0}^\infty G_m \exp \left( -\frac{2 \lambda_m^2 x^*}{3} \right)
\]

(12.5.17)

where the first three coefficients are:

<table>
<thead>
<tr>
<th>( m )</th>
<th>( G_m )</th>
<th>( \lambda_m^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.910</td>
<td>2.83</td>
</tr>
<tr>
<td>1</td>
<td>0.0533</td>
<td>32.1</td>
</tr>
<tr>
<td>2</td>
<td>0.0153</td>
<td>93.4</td>
</tr>
</tbody>
</table>

A one-term approximation to the infinite series solution, valid for values of \( x^* > 0.1 \), is

\[
\Theta_{cm} = 0.91 \exp (-1.89x^*)
\]

(12.5.18)

Everything we need to know about the average temperature of the fluid is given in the solution for \( \Theta_{cm} \). (For purposes of design or analysis of an existing system, the essential information is found in this cup-mixing average, rather than in the detailed temperature distribution \( \Theta(x^*, y^*) \).) It is useful at this point to calculate a measure of the rate of transfer of heat across the isothermal boundary, and we usually do this by first defining a heat transfer coefficient. The local convective heat transfer coefficient \( h_\text{in} \) is defined in terms of a heat balance on the fluid:

\[
-\rho C_p UH dT_{cm} = h_\text{in} (T_{cm} - T_H) \, dx
\]

(12.5.19)

Note that in this expression the heat transfer coefficient is based on the difference between the average temperature in the fluid (and it is the cup-mixing average) and the wall temperature. Note also that Eq. 12.5.19 is for the transfer across the lower half of the channel. For both sides of the channel, we would simply double each side of this expression.

Then we may convert to the dimensionless format and calculate the heat transfer coefficient from

\[
h_\text{in} = \left( \frac{UH}{\Theta_{cm}} \right) \left( \frac{k}{UH^2} \right) \left( \frac{d\Theta_{cm}}{dx^*} \right)
\]

(12.5.20)

(We have changed from \( dx \) to \( dx^* \) in this expression, as well as from \( T \) to \( \Theta \).)
Now we must go back to the solution for $\Theta_{cm}$ (Eq. 12.5.17) and differentiate $\Theta_{cm}$ with respect to $x^*$. The result may be written in the format

$$\frac{4h_{ln}H}{k} = \frac{\sum_{m=0}^{\infty} \lambda_m^2 G_m \exp(-2\lambda_m^2 x^*/3)}{\sum_{m=0}^{\infty} G_m \exp(-2\lambda_m^2 x^*/3)}$$  \hspace{1cm} (12.5.21)

where the Nusselt number is based on the length scale $4H$. However, this is a local Nusselt number, since the function on the right-hand side is $x^*$ dependent. For large enough values of $x^*$, we may use a one-term approximation to the infinite series that appear in the numerator and denominator, with the result that the Nusselt number approaches a constant value given by

$$Nu_{ln} = \lambda_0^2 = 7.55$$  \hspace{1cm} (12.5.22)

Again, with all the algebra flying around, we need to remind ourselves of what we have derived. Equation 12.5.22 is valid only for distances $x^*$ that are in some sense far downstream from the entrance to the heat exchanger. By carrying out a more detailed analysis, we could show that this is a good approximation as long as $x^*$ exceeds a value of order 0.1. Keep in mind that this $x^*$ is a dimensionless length. Whether this simple model is useful depends on whether most of the heat exchanger lies in the region $x^* > 0.1$, and of course this depends on the parameters $U$ and $\alpha$, especially.

We will find soon that we often need the averaged value of the Nusselt number more than the local value, in design problems. The average is defined as

$$\overline{Nu}_L = \frac{1}{x_L^*} \int_0^{x_L^*} Nu_{ln}(x^*) \, dx^*$$  \hspace{1cm} (12.5.23)

where $x_L^*$ is simply $x^*$ (Eq. 12.5.9) at $x = L$. Looking at Eq. 12.5.21 we see that we would have to evaluate the integral of a ratio of two infinite series. A much simpler algebraic form results if we use an energy balance approach, and this is left as a homework exercise (Problem 12.17). The result is that the mean Nusselt number can be calculated quite simply from

$$\overline{Nu}_L = \frac{4}{x_L^*} \ln \left( \frac{1}{\Theta_{cm}(x_L^*)} \right)$$  \hspace{1cm} (12.5.24)

This is the mean that has been averaged over the length $x_L^*$.

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1 We adopt the common convention that uses the "hydraulic diameter" as the length scale in the Nusselt and Reynolds numbers. The hydraulic diameter is defined, for a tube of any cross-sectional shape, as

$$D_h = \frac{4 \times \text{cross-sectional area}}{witted \text{ perimeter}}$$

For the circular cross section this gives the expected result

$$D_h = \frac{4(\pi D^2/4)}{\pi D} = D$$

For parallel plates of width $W$ spaced a distance $2H$ apart, we find

$$D_h = \frac{4(2H/W)}{2W + 4H} = 4H \quad \text{for} \quad W/H \gg 1$$
12.5.2 The Thermal Entry Region of a Parallel Plate Heat Exchanger with Isothermal Surfaces

Although Eq. 12.5.17 is applicable for small values of $x^*$, it is not a computationally efficient model because of the need to sum over a very large number of terms when $x^*$ is small. Since we would like to avoid the work of evaluating a large number of the $G_m$ and $\lambda_m$ coefficients, we shall examine a model that is useful for very short exchangers. It turns out that we can find a solution to Eq. 12.5.3 in a very simple form that is valid only for small values of $x^*$.

We begin, with reference to Fig. 12.5.2, by considering a region of small $x$ such that the temperature change in the fluid is confined to a region that is close to the wall, in comparison to the separation ($2H$) between the plates. Beyond the point $x = 0$, the surface is at the temperature $T = T_H$, and a temperature profile develops as heat is conducted into the flowing stream. In developing a model for the entrance region of the exchanger, we assume that the change in the temperature field is confined to a region close enough to the solid surface to permit the approximation of the velocity profile as a linear function of $y$. We will see why we want to assume this in a moment.

We now want to solve

$$u(y) \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{12.5.25}$$

with the following boundary conditions:

$$T = T_1 \quad \text{at } x = 0 \quad \text{for all } y \tag{12.5.26}$$

$$T = T_H \quad \text{at } y = 0 \quad \text{for all } x \geq 0 \tag{12.5.27}$$

(In comparison to the preceding analysis, we have shifted the $y$ axis so that the lower surface is at $y = 0$ and the central plane is at $y = H$.) Otherwise, these are the same boundary conditions applied earlier, and there is no reason to change them. It is the final boundary condition that we change. Although we still have symmetry about the central plane $y = H$, we no longer write the boundary condition in that form.

For a third boundary condition we will assume that the penetration of heat is small in comparison to some length scale of the flow field in the $y$ direction. Really, we require that the penetration length be small compared to the distance over which the velocity profile changes, because we are going to assume that the velocity profile is linear throughout the penetration region (i.e., the distance in the $y$ direction over which $T$ changes from $T_H$ to $T_1$). This assumption will put Eq. 12.5.25 in a form that yields an analytical solution useful for computational purposes. That form is

$$\beta y \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{12.5.28}$$

\[\text{Figure 12.5.2 Heat transfer in the entrance region of a parallel plate exchanger. The second plate, which is parallel to the one shown, is in the plane } y = 2H.\]
where $\beta$ is simply the velocity gradient at $y = 0$ (the solid surface). This follows because we have linearized the velocity profile (Eq. 12.5.1) by writing

$$u_x = u_x(y = 0) + \left[ \frac{\partial u_x}{\partial y} \right]_{y=0} y + \cdots \approx \beta y$$  \hspace{1cm} (12.5.29)

where

$$\beta = \frac{3U}{H}$$  \hspace{1cm} (12.5.30)

The third boundary condition is then written as

$$T = T_1 \quad \text{for} \quad y = \infty$$  \hspace{1cm} (12.5.31)

This reflects the notion that if the penetration region is very small, there will be no measurable change in temperature in the central core of the fluid, and the system will behave with respect to heat transfer as if it were infinite in the $y$ direction.

This “boundary value problem” may be solved by the method of combination of variables, and the result (see Section 5.4 of the Mass Transfer section for the details, where the same mathematical model arises) is

$$\Theta = 1 - \frac{\int_0^\infty \exp(-\eta^2) \ d\eta}{\int_0^\infty \exp(-\eta^2) \ d\eta}$$  \hspace{1cm} (12.5.32)

The dimensionless temperature $\Theta$ is defined as before (Eq. 12.5.8). The space variables are handled in a very different way now, because of our assumption that the fluid is infinite in the $y$ direction. In a sense, there is no longer a length scale $H$ in this formulation of the problem. The space variables have been combined into a single grouping, $\eta$, defined as

$$\eta = y \left( \frac{\beta}{9\alpha x} \right)^{1/3}$$  \hspace{1cm} (12.5.33)

The definite integral in the denominator of Eq. 12.5.32 is a pure number. (It is a so-called gamma function, in this case $\Gamma(4/3)$, and has the value $\Gamma(4/3) = 0.893$.

The local heat flux along the surface $y = 0$ is found from

$$q_y(x) = -k \frac{\partial T}{\partial y} \bigg|_{y=0} = k(T_H - T_1) \left( \frac{\beta/9\alpha x}{\Gamma(4/3)} \right)^{1/3}$$  \hspace{1cm} (12.5.34)

Per unit width $W$, the total rate at which heat is transferred across a length $L$ of a surface is

$$\frac{Q_H}{W} = -k \int_0^L \frac{\partial T}{\partial y} \bigg|_{y=0} \ dx = \frac{3k(T_H - T_1)}{2\Gamma(4/3)} \left( \frac{\beta L^3}{9\alpha} \right)^{1/3}$$  \hspace{1cm} (12.5.35)

We may use Eq. 12.5.34 to define and calculate a local heat transfer coefficient as

$$h(x) = \frac{q_y(x)}{T_H - T_1} = k \left( \frac{\beta/9\alpha x}{\Gamma(4/3)} \right)^{1/3}$$  \hspace{1cm} (12.5.36)

With $\beta$ from Eq. 12.5.30, we may ultimately write this equation in the form

$$\text{Nu}(x) = \frac{4h(x)H}{k} = \frac{4(UH^2/3\alpha x)^{1/3}}{\Gamma(4/3)} = 3.12(x^*)^{-1/3}$$  \hspace{1cm} (12.5.37)
where $x^*$, defined as in Eq. 12.5.9, is $x^* = (4x/H)/\text{Re Pr}$, and the Reynolds number is based on the length scale $4H$:

$$\text{Re} = \frac{4UH}{\nu} \quad (12.5.38)$$

Equation 12.5.37 gives the local Nusselt number. For design purposes it is often useful to have a Nusselt number averaged over the length $L$. A useful definition of an averaged $h$ is

$$\bar{h}_L = \frac{Q_H/W}{L(T_H - T_1)} = \frac{3k}{2\Gamma(4/3)} \left( \frac{\beta}{9\alpha L} \right)^{1/3} \quad (12.5.39)$$

and the corresponding averaged Nusselt number is

$$\bar{\text{Nu}}_L = \frac{4\bar{h}_L H}{k} = 2.95 \left( \frac{\text{Re Pr}}{L/H} \right)^{1/3} = 4.425 x_L^{-1/3} \quad (12.5.40)$$

where

$$x_L^* = \left( \frac{\text{Re Pr} H}{4L} \right)^{-1} \quad (12.5.41)$$

To find the averaged temperature at some position down the length of the exchanger, we could use Eq. 12.5.32, but it is easier to work with the heat balance. We have $Q_H$ in Eq. 12.5.39, since

$$\frac{Q_H}{W} = \rho C_p(T_{cm} - T_1) q_w = \bar{h}_L L(T_H - T_1) = \frac{3k}{2\Gamma(4/3)} (\frac{\beta L^2}{9\alpha})^{1/3} \quad (12.5.42)$$

Since Eq. 12.5.42 gives the heat transferred from half of the system (the lower plate), we use half the flowrate per unit width: $q_w/2 = UH$. When we rearrange Eq. 12.5.42 and identify and solve for $\Theta_{cm}$, we find

$$\Theta_{cm} = 1 \frac{T_{cm} - T_1}{T_H - T_1} = 1 - \frac{k}{UH \rho C_p 2\Gamma(4/3)} \left( \frac{\beta L^2}{9\alpha} \right)^{1/3} \quad (12.5.43)$$

which we may write, after some further rearrangement, in the form

$$\Theta_{cm} = 1 - 2.95 \left( \frac{\text{Re Pr} H}{L} \right)^{-2/3} = 1 - 1.17 (x_L^*)^{2/3} \quad (12.5.44)$$

**Figure 12.5.3** Cup-mixing temperature along the axis of a parallel plate, laminar flow heat exchanger.
Figure 12.5.4 Local Nusselt numbers along the axis of a parallel plate, laminar flow heat exchanger. Both surfaces at the same constant temperature. Also shown is the averaged Nusselt number plotted against $x^*; D_h = 4H$.

Figure 12.5.3 shows the variation of the cup-mixing (dimensionless) temperature along the axis of the channel. Equation 12.5.44 is valid only for small values of $x^*$, since the primary restriction on this model is that the degree of penetration of the wall temperature into the fluid must lie within a region so close to the surface that the linearization of the velocity profile is a good approximation. We also show the predicted temperature from Eq. 12.5.17, which does not have that restriction. However, Eq. 12.5.17 is not accurate unless we use a large number of terms of the infinite series solution, for $x^*$ below 0.1. Hence these two solutions cover the range of small and large $x^*$.

While Fig. 12.5.3 provides a graphical means of calculating the cup-mixing temperature, and this is normally sufficient information to solve a heat exchanger design problem, it is interesting to examine the local heat transfer coefficient for the parallel plate system. This is shown in Fig. 12.5.4. Again, we use the infinite series solution (but just with a few terms) for large $x^*$, and we use Eq. 12.5.35 for small values of $x^*$. 