Quantum Mechanics postulates:

(For an isolated system)

Postulate 1: Associated to any physical system is a Hilbert space, an isolated system is completely described by a unit vector in this Hilbert space. \( |\Psi\rangle \)

Two-dim : qubit \( |\Psi\rangle = a|0\rangle + b|1\rangle \)

\( \langle \Psi | \Psi \rangle = 1 \rightarrow |a|^2 + |b|^2 = 1 \)

Difference between a bit & qubit \( \rightarrow \) superposition.

Wave function: \( |\Psi\rangle = \int \psi(x) \langle x | \Psi \rangle \ dx \)

\( \psi(x) = \langle x | \Psi \rangle \)
Evolution: (postulate 2)

- The state of a closed system can evolve in time described by unitary transformation

\[ |y'\rangle = U |y\rangle \]

\( U \) is dictated by the physics.

- norm is preserved

\[ \langle y'|y'\rangle = \langle y|U^\dagger U |y\rangle \]

postulate 2': The time evolution is described by Schrödinger equation:

\[ i\hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle \]

\( 2\pi\hbar = h_\text{p} \): plank's constant: 6.62 x 10^{-34} \text{ m}^2\text{kg/s} 

\( H \): Hamiltonian. determined by the interactions.

energy operator \( (H = \sum E_k \chi_k \chi_k^\dagger) \) → stationary state
Solution of the Schrödinger eq.

\[ \frac{-i \hbar}{\hbar} \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle \to |\psi(t)\rangle = e^{\frac{-iHt}{\hbar}} |\psi(0)\rangle \]

Unitary transformation.

classical mechanics: \( F = ma \to m \ddot{\mathbf{r}} = -\frac{dV}{dt} \)

\[ E_{\text{tot}} = \frac{1}{2} m \dot{\mathbf{r}}^2 + V(\mathbf{r}) \quad \ddot{\mathbf{r}} = \frac{2}{m} \sqrt{E_{\text{tot}} - V} \]

potencial energy

Postulate 3: To each dynamical variable (physical concept) there corresponds a Hermitian operator and the possible values of the dynamical variable are the eigenvalues of the operator.
Hamiltonian $H$: Energy operator. (also $\hat{E} = i\hbar \frac{\partial}{\partial t}$)

classical mechanics: $E_{\text{tot}} = \frac{1}{2m} \hat{p}^2 + V(x)$

Total energy of a single particle: $E_{\text{tot}} = \frac{1}{2m} \hat{p}^2 + V(x)$

Quantum: $H = \frac{1}{2m} \hat{p}^2 + V(\hat{x}) = \hat{p} \hat{x}$ are momentum and position operators.

Wave Function:

$$+i\hbar \psi' = (\frac{\hat{p}^2}{2m} + V(\hat{x})) \psi$$

$$\int dx (-i\hbar \nabla \psi) = \int \text{d}x \left( i\hbar \nabla \hat{x} \right) \left( \frac{\hat{p}^2}{2m} + V(\hat{x}) \right) \psi$$

The momentum operator in position basis:

$$\hat{p} = \hat{p} - i\hbar \frac{\partial}{\partial x}$$

Define the wave function $\Phi(x) = \langle x | \psi \rangle$

$$\implies +i\hbar \nabla \psi = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \Phi(x)$$
Particle in a 1-D box:

\[
\psi(x) = \begin{cases} 
0, & 0 < x < L \\
\infty, & |x| > L 
\end{cases}
\]

Hilbert space \( \mathcal{H} \): Space of quadratically integrable functions, \( L^2 \)

\( f(x) \) is quadratically integrable iff \( \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \)

Inner product: \( \langle f | g \rangle = \int_{-\infty}^{\infty} f(x)g(x) \, dx \)

Above example continued:

General solution: \( [A \sin(\kappa x) + B \cos(\kappa x)] e^{-i\omega t} \)

\( E = \hbar \omega + \frac{\hbar^2 \kappa^2}{2m} \)

Boundary condition: \( \psi(x) \to 0 \) outside the well.

\( \psi_n(x, t) = \begin{cases} 
A \sin(\kappa_n x) e^{-i\omega t}, & 0 < x < L \\
0, & |x| > L 
\end{cases} \)

\( \kappa_n = \frac{n\pi}{L} \)
Normalized wavefunction \[ \int_0^\infty |\psi(x)|^2 \, dx = 1 \]

\[ |A| = \sqrt{\frac{2}{L}} \]

Discrete energy levels: \[ E_n = \frac{n^2 \hbar^2 \pi^2}{2mL^2} \]

Ground state: \( n=1 \)
Quantum measurement:

Stern-Gerlach experiment (1921).

Magnetic moment:

\[ M = I A = \frac{ev}{2ar} \cdot ar^2 = \frac{1}{2} evr = \frac{e}{2m} L \quad (L = mv) \]

Bohr theory of atoms: \( L \) is quantized, \( L = nh \)

\[ n = 1, 2, \ldots \]

Force experienced by the dipole moment:

\[ F = -\frac{d}{dz} (M \cdot B) = -\frac{e}{2m} L^2 \oint dB \frac{dz}{dz} \]

An scheme to measure \( L_2 \):

classical physics:

\[ \rightarrow \text{a continuous distribution}. \]
observation with different atoms.

The actual experiment reveals that the atoms are deflected & splitted in a discrete set of outcomes.

Depends on the atom type $L_z$ can take some integer number between $-l & l$ for some $l \epsilon \mathbb{N}$.

For silver, hydrogen, copper, gold -- $l = 0$ -- therefore we expect no deflection of the atom beam.

$\Rightarrow$ Discovery of electron spin: Internal or inherent dipole moment.

With silver atom Stern & Gerlach observed two spots

\[ L_z(l=1/2) = S \{ +1/2, -1/2 \} \]
Goal was to measure $L_2$ but the outcomes are just a discrete set $\Rightarrow$ projective measurement.

Postulate 4: (projective measurement)

Quantum measurements are described by a collection of projection operators $\{P_m\}$. The set of indices $\{m\}$ represents the set of outcomes.

If the pre-measurement state of the system is $|\psi\rangle$

After measurement its state will be $\overline{P_m|\psi\rangle} = \frac{<\psi|P_m|\psi>}{<\psi|\psi>}$

with probability $<\psi|P_m|\psi> = P(m)$

$\sum P(m) = 1 \Rightarrow \sum P_m = I.$

Stern-Gerlach: $P_{+\frac{1}{2}} = |+\frac{1}{2}\rangle X |+\frac{1}{2}\rangle$, $P_{-\frac{1}{2}} = \frac{1}{2} X - \frac{1}{2} |$
\[ |\Psi\rangle = a | +1/2 \rangle + b | -1/2 \rangle \]

\[
P( +1/2 ) = \langle \Psi | P_{ +1/2 } | \Psi \rangle = |a|^2
\]

\[
P( -1/2 ) = \langle \Psi | P_{ -1/2 } | \Psi \rangle = |b|^2
\]

\[ \rightarrow \text{Expectation value of the observable} \quad S : \]

\[
E(S) = \frac{1}{2} + \frac{1}{2} P( +1/2 ) + (-1/2) P( -1/2 ) = \frac{|a|^2 - |b|^2}{2}
\]

\[ \rightarrow \text{Result} \quad E(S) = \langle \Psi | S | \Psi \rangle \quad \text{where} \quad S = \begin{pmatrix} +1/2 & 0 \\ 0 & -1/2 \end{pmatrix}
\]

\[ \text{Standard deviation of} \quad S \quad \text{measurement:} \]

\[ \sqrt{\langle \Psi | S^2 | \Psi \rangle - \langle \Psi | S | \Psi \rangle^2} \]

Example 2: photon polarization measurement

A propagating electromagnetic field has two polarization.
measuring the polarization of a single photon:

beam splitter

\[ \text{photon detector} \]

The incoming photon has a polarization state \( |H \rangle + e^{i\theta} |V \rangle \)

after detection the state will be either \( |H \rangle \) or \( |V \rangle \)

though the photon will be destroyed at the end.

(A bit tricky)

The wavefunction collapses after a measurement process.

This shows the intrinsic difference between quantum and classical measurement processes.
General set-up to measure an observable $A$:

$$A = \sum_a a |a \rangle \langle a|,$$

Design a setup with outcomes $|a\rangle$.

Since the measurement is a probabilistic process useful information can be obtained through some statistical formalism.

Repeat the measurement for an ensemble of $N$ identical copies of the same state $|\psi\rangle$.

Expectation value of observable $(A)$

$$E(A) = \sum_a a P(a) = \sum_a a \langle \psi | a \rangle \langle a | \psi \rangle = \sum_a |\langle a | \psi \rangle|^2.$$

Standard deviation:

$$\Delta(A) = \sum a^2 P(a) - (\sum a P(a))^2 = \langle \psi | A^2 | \psi \rangle - \langle \psi | A | \psi \rangle^2.$$
Heisenberg Uncertainty principle:

\[ |\langle \psi | [A, B] |\psi \rangle|^2 + |\langle \psi | A, B |\psi \rangle|^2 = 4 |\langle \psi | AB |\psi \rangle|^2 \]

Cauchy-Schwartz inequality:

\[ |\langle \psi | AB |\psi \rangle|^2 \leq \langle \psi | A^2 |\psi \rangle \langle \psi | B^2 |\psi \rangle \]

\[ \Rightarrow \quad |\langle \psi | A, B |\psi \rangle|^2 \leq 4 \langle \psi | A^2 |\psi \rangle \langle \psi | B^2 |\psi \rangle \]

If \( A = C - \langle c \rangle \) & \( D = D - \langle D \rangle \)

\[ \Rightarrow \quad \Delta(c) \Delta(D) \geq \frac{1}{2} |\langle \psi | C, D |\psi \rangle| \]

Interpretation: If we measure observables \( C \) & \( D \)
on two similar ensembles of state \( |\psi \rangle \) thestandard deviation of \( C \) & \( D \) satisfy the aboverelation.
Uncertainty in position & momentum of a quantum particle:

\[ \Delta x \Delta \hat{p} \geq \frac{1}{2} \left\langle \hat{p}^2 \hat{x} \right\rangle \]

\[ [\hat{p}, \hat{x}] = i \hbar \phi_x = \left( \frac{\partial}{\partial x} \right) \left( x \phi_x \right) - x \left( \frac{\partial}{\partial x} \phi_x \right) \]

\[ = i \hbar \phi_x \quad \Rightarrow 

[\hat{p}, \hat{x}] = i \hbar \]

\[ \Rightarrow \Delta x \Delta \hat{p} \geq \frac{\hbar}{2} = \frac{\hbar}{4\pi} \]

In contrast to classical physics, quantum mechanics describe a particle as it cannot have both a definite position & a definite momentum.

Uncertainty principle is a consequence of wave-particle duality.
Uncertainty principle in signal processing:

Spectrum of a signal \( \phi(t) \):

\[
\hat{\phi}(\omega) = \int e^{-i2\pi\omega t} \phi(t) \, dt
\]

\[
\phi(t) = \int \hat{\phi}(\omega) e^{i2\pi\omega t} \, d\omega
\]

U. P.: A signal cannot be localized both in time & frequency.

\[ \Delta t \Delta \omega \geq \frac{1}{2} \]

Composite systems or many degrees of freedom:

Postulate 5: The state space of a composite physical system is the tensor product of the state spaces of the component physical systems.
Two quantum states that are different up to a global phase are equivalent:

\[ |\Psi\rangle \sim e^{i\theta} |\Psi\rangle = |\Phi\rangle \]

Why? Because a global phase does not change the expectation value of any observable:

\[ \langle \Phi | \Omega | \Phi \rangle = \langle \Psi | e^{-i\theta} e^{i\theta} | \Psi \rangle = \langle \Psi | \Omega | \Psi \rangle \]

Physical significance: The exact value of the ground state energy of a system is not important.

\[ H \rightarrow H + \epsilon I \] : A constant shift of Hamiltonian

\[ |\Psi(t)\rangle = e^{-i(H+\epsilon I)t} |\Psi(0)\rangle = e^{-i\epsilon t} e^{-iHt} |\Psi(0)\rangle \]

just a global phase.
relative phase:

\[ 1\Phi_2 \]

\[ 1\Phi_1 = \frac{10\rangle + 11\rangle}{\sqrt{2}} \quad \text{and} \quad \frac{10\rangle - 11\rangle}{\sqrt{2}} \]

have the same amplitude in the measurement basis \(10\rangle & 11\rangle\).

We cannot distinguish these two states by measuring operators \(10\langle 01 \& 11\langle 11\rangle\).

But we can do it by projecting them on \(1\Phi_1 \times \Phi_1\) and \(1\Phi_2 \times \Phi_2\), since \(\langle 4\rangle |\Phi_2 \rangle = 0\).

What makes \(1\Phi_1\) different from \(1\Phi_2\) is the relative phase difference \(\pi/2\):

\[
\begin{cases}
1\Phi_1 = \frac{10\rangle + 11\rangle}{\sqrt{2}} \\
1\Phi_2 = \frac{10\rangle + e^{-i\pi/2} 11\rangle}{\sqrt{2}}
\end{cases}
\]
Two particles: 

$|i\rangle_a \otimes |k\rangle_b$; orthonormal basis.

$L\phi_1\rangle = \sum \Psi_{ik} |i\rangle_a \otimes |k\rangle_b \neq \phi_1\rangle A \otimes \phi_1\rangle B$

The composite basis: $|i\rangle_a \otimes |k\rangle_b \leftrightarrow \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$

This construction is not only for two separate particles but for two independent physical entities of the same particle. For instance, the Hilbert space of an electron is defined by $\mathcal{H} \otimes \mathcal{H}$ spatial spin.

Entangled state: $\phi_{AB} \neq \phi_A \otimes \phi_B$

If a state cannot be written as a product of two state.
Density matrix:

what is the state of a subsystem:

One particle from a two-particle setting.

\[ |\psi_{12}\rangle = \sum_{i_1, i_2} a_{i_1 i_2} |i_1\rangle |i_2\rangle \]

Local observable: \( O_1 \otimes I_2 \rightarrow \) identity

\[ \langle O_1 \otimes I_2 \rangle = \sum_{i_1, i_2} \langle \psi_{12} | O_1 \otimes I_2 | \psi_{12} \rangle = \sum_{i_1, i_2, j_1, j_2} a_{i_1 i_2}^* a_{j_1 j_2} \langle i_1 | O_1 | i_2 \rangle \langle i_2 | I_2 | j_2 \rangle \]

\[ = \sum_{i_1, i_2, j_1} a_{i_1 i_2}^* a_{j_1 j_2} \langle i_1 | O_1 | i_2 \rangle \langle i_2 | I_2 | j_1 \rangle \]

\[ = \text{Tr} (O_1 \sum_{i_1, i_2} a_{i_1 i_2}^* a_{j_1 j_2} |i_2 \rangle \langle j_1|) \]

\[ = \text{Tr} (O_1 S_1) \]

density matrix \( O_1 : S_1 = \sum_{i_1, i_2} a_{i_1 i_2}^* a_{j_1 j_2} |i_2 \rangle \langle j_1| = \text{Tr} (|\psi_{12}\rangle \langle \psi_{12}|) \) partial trace
principles of quantum mechanics for open (non-isolated) systems:

If two systems A & B have never interacted then and

their initial state has been \( |\psi_{AB}\rangle = |\psi_A\rangle |\psi_B\rangle \)

Then at any time \( T \) their state will remain a

product state: \( |\psi_{AB}(T)\rangle = |\psi_A(T)\rangle \otimes |\psi_B(T)\rangle \)

\[
\Rightarrow S_A(T) = \text{Tr}_B ( |\psi_{AB}(T)\rangle \langle \psi_{AB}(T)| ) \\
= |\psi_A(T)\rangle \langle \psi_A(T)|
\]

\( \Rightarrow \) The state of system A is given by \( |\psi_A(T)\rangle \).

What if A & B interact? A is not isolated anymore.
Quantum Mechanics postulates:

QM is a mathematical framework for the development of physical theories.

1- State space: Associated to any physical system is a Hilbert space. The system state is completely described by a positive trace one operator acting on this Hilbert space. This operator is called density matrix.

- Classical mechanics: The state of a particle is specified by six real parameters \( \{x, y, z, p_x, p_y, p_z\} \); a vector in a Euclidian space.
properties of density matrix:

1. $\rho$ is a positive definite matrix: $\rho > 0$.
2. $\text{Tr}(\rho) = 1$: Guarantees total probability 1.

Traditional meaning of density matrix:

For an ensemble of quantum systems $\mathcal{H}$ of an ensemble set $\{ |\psi_i\rangle \}$.

If the total portion of systems in state $|\psi_i\rangle$ is $p_i$, then the total state of this ensemble is described by a density matrix: $\rho = \sum p_i |\psi_i\rangle \langle \psi_i|$.

With the advance of technology, engineering of a single quantum system became possible and the wavefunction description of a quantum system is not enough was
Any density matrix has a full spectral decomposition:

$$\rho = \sum \lambda_i |i\rangle \langle i|, \quad \lambda_i \geq 0 \quad \text{and} \quad \sum \lambda_i = 1.$$ 

A rank one density matrix: pure state, \( \rho = |i\rangle \langle i| \).

Postulate 2: The evolution of a closed quantum system is given by a unitary transformation:

$$\rho' = U \rho U^\dagger.$$ 

Time evolution: \( \dot{\rho} = -i [H, \rho] \) (Liouville equation).

Postulate 3: (Projective measurement) described by a set of projectors \( \{P_i\} \).

The outcome of measurement will be \( i \)th with probability \( P(i) = \text{Tr}(P_i \rho) \) and the state collapses to \( \rho_i = \frac{P_i \rho P_i}{P(i)} \).
Postulate 4: The state space of a composite physical system is the tensor product of the state space of the component physical system.

Operational meaning of density matrix (HW?)

Different ensembles can give rise to the same density matrix.

\[ \rho = \frac{3}{4} |00\rangle \langle 00| + \frac{1}{4} |11\rangle \langle 11| \]

\[ = \frac{1}{2} |a\rangle \langle a| + \frac{1}{2} |b\rangle \langle b| \]

\[ |a\rangle = \sqrt{\frac{3}{4}} |0\rangle + \sqrt{\frac{1}{4}} |1\rangle \]

\[ \langle a | b \rangle \neq 0 \]

\[ |b\rangle = \sqrt{\frac{3}{4}} |0\rangle - \sqrt{\frac{1}{4}} |1\rangle \]
Block sphere Representation:

For a single qubit: \( s = \frac{1}{2} (I + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z) \)

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\]

\[
\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

\( r_x, r_y, r_z \in \mathbb{R}, \quad r_x^2 + r_y^2 + r_z^2 \leq 1 \)

Example of qubits: electron spin

electron position in a two-well potential (quantum dot)

photon polarization
Spin in a constant magnetic field:

\[ H = \frac{e}{m} \mathbf{B} \cdot \mathbf{S} \]

\[ \mathbf{S} = \frac{\hbar}{2} \left( \delta_x \delta_y \right) \]

\[ \Rightarrow H = \frac{e}{m} \mathbf{B} \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right) \]

\[ i \hbar \dot{S} = \hbar \left[ H, \mathbf{S} \right] \Rightarrow i \hbar \left( r_x \delta_y \mathbf{y} + r_y \delta_x \mathbf{y} \right) \]

\[ = \hbar \omega \left( r_x (2i) \delta_y - r_y (2i) \delta_x \right) \]

\[ \Rightarrow \]

\[ \begin{align*}
\dot{r}_x &= -2 \omega r_y \\
\dot{r}_y &= 2 \omega r_x \\
\end{align*} \]

\[ r_x^2 + r_y^2 = \text{constant} \]

\[ r_x \text{ and } r_y \text{ oscillate with } \]

\[ \text{larger frequency } \frac{eB}{m}. \]