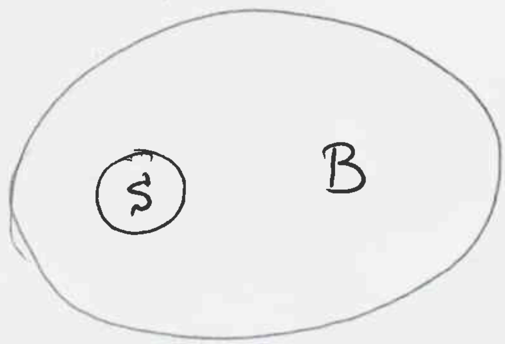
Open quantum systems:

The environment is a set of degrees of freedoms that can be considered as a second quantum system coupled to the main system.



S: System

B: Bath or environment.

$$\frac{d}{dt}\rho_{SB}(t) = -i\,[H_{SB}, \rho_{SB}(t)]$$

$$\Longrightarrow \rho_{SB}(t) = e^{-iH_{SB}t}\rho_{SB}(0)\,e^{iH_{SB}(t)} \qquad (\hbar = 1)$$

$$\rho_S(t) = \mathrm{Tr}_B(\rho_{SB}(t)) = \mathrm{Tr}_B\left(e^{-iH_{SB}t}\rho_{SB}(0)\,e^{iH_{SB}t}\right)$$

If $\rho_{SB}(0) = \rho_S \otimes \rho_B$

and $\rho_B = \sum_\lambda \lambda |b_\lambda\rangle\langle b_\lambda|$

$$\Rightarrow \rho_S(t) = \sum_{\lambda,\mu} \sqrt{\lambda} \langle b_\mu | U_t | b_\lambda \rangle \rho(0) \langle b_\lambda | U_t^\dagger | b_\mu \rangle \sqrt{\lambda}$$

$$\boxed{U_t = e^{-iH_{SB}t}} \quad \Rightarrow \rho_S(t) = \sum_k E_k(t) \rho_S(0) E_k^\dagger(t)$$

$$E_k(t) = \sqrt{\lambda} \langle b_\mu | U_t | b_\lambda \rangle$$

$$k = (\lambda, \mu)$$

(Kraus-Representation or Completely positive map)

A cp map Φ not only preserves the positivity of the input density matrix $\Phi_S(\rho_S) \geq 0$

but also any extended space $\Phi_S \otimes I_{S'}(\rho_{SS'}) \geq 0$

Trace preservation: $\sum E_k^\dagger E_k = 1$.

Choi's Theorem:

A Linear map Φ is CP iff

$$\sum_{i,j} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|) \geq 0$$

Single qubit dephasing channel:

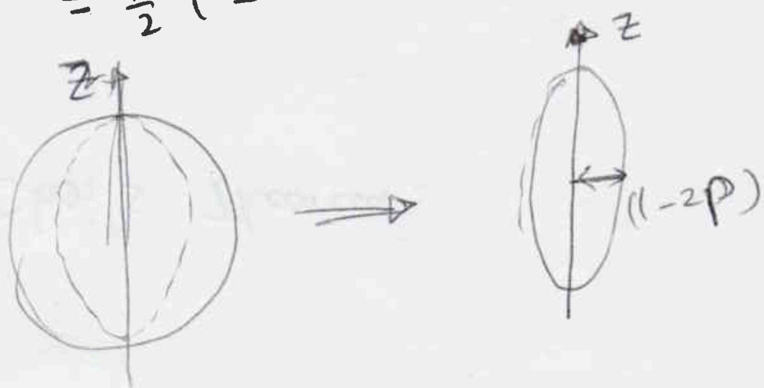
$$\mathcal{E}(\rho) = (1-p)\rho + p\sigma_z \rho \sigma_z$$

p: probability of error σ_z.

$$\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$$

$$\mathcal{E}(\rho) = (1-p)\rho + p/2\,(I - r_x X - r_y Y + r_z Z)$$

$$= \frac{1}{2}\left(I + (1-2p) r_x X + (1-2p) r_y Y + r_z Z\right)$$



Loss of phase through random kicks:

$|\psi\rangle = a|0\rangle + b|1\rangle$ rotate by the stochastic Hamiltonian $H = \theta\sigma_z$ where θ is a random ~~variable~~ variable with a Gaussian distribution.

$$\text{Averge } \rho = \frac{1}{\sqrt{4\pi\lambda}} \int_{-\infty}^{\infty} R_z(\theta)|\psi\rangle\langle\psi| R_z^{\dagger}(\theta)\, e^{-\theta^2/4\lambda}\, d\theta$$

where $R_z(\theta) = e^{-i\theta Z}$ $\Longrightarrow$ $\rho = \begin{pmatrix} |a|^2 & ab^* e^{-\lambda} \\ a^* b e^{-\lambda} & |b|^2 \end{pmatrix}$

The off-diagonals decay exponetially fast to zero.

Dephasing : Loss of phase

~~$|\psi\rangle = |a\rangle$~~

$$|\psi\rangle = a|0\rangle + be^{i\phi}|1\rangle \Longrightarrow \rho = \begin{pmatrix} |a|^2 & ab^* e^{-i\phi} \\ a^* b e^{i\phi} & |b|^2 \end{pmatrix}$$

after dephasing $\Longrightarrow$ $\begin{pmatrix} |a|^2 & \frac{ab^*}{2}(1-2p)e^{-i\phi} \\ \frac{a^* b}{2}(1-2p)e^{i\phi} & |b|^2 \end{pmatrix}$

ultimately $\longrightarrow$ $\begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix}$: Loss of information. (Quantum)

Damping channel: (Loss of energy)

~~$\mathcal{E}(\rho) = (1-p)\rho + p\sigma_-\rho\sigma_+$~~

$$\boxed{\begin{aligned} \sigma_- &= |0\rangle\langle 1| \\ \sigma_+ &= \sigma_-^\dagger \end{aligned}}$$

$$\mathcal{E}(\rho) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \rho \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} + p\sigma_-\rho\sigma_+$$

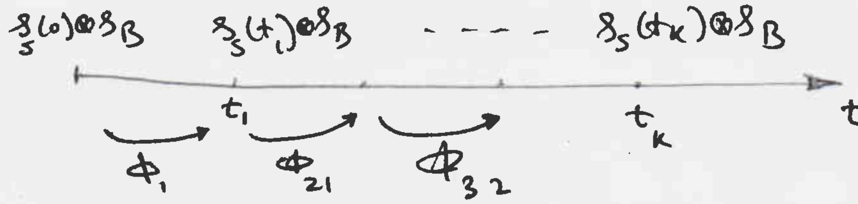
$$\mathcal{E}(\rho) = \begin{pmatrix} 1-(1-p)(1-\rho_{11}) & \rho_{12}\sqrt{1-p} \\ \rho_{12}^{*}\sqrt{1-p} & \rho_{22}(1-p) \end{pmatrix}$$

Quantum Dynamical Equation:

Is there a compact form for the time variations of the density matrix? $\frac{d\rho}{dt} = ?$

A differential equation can be obtained under certain conditions & approximations.

The most well-known & mostly used / applicable condition is a large & fast bath, that yields a Markovian behavior of the system.



The bath is rigid & large : $\rho_{SB}(t) \approx \rho_S(t) \otimes \rho_B$

$\Rightarrow$ all mappings between times (t_i, t_j) is a cp map due to the assumption that at each initial time t_i system & bath is in a product state.

$$\rho_{t+dt} = \Phi_{(t,t+dt)}\, \rho_t$$

$$\Phi_{t,t+dt} = \mathbb{1} + dt\, L$$

for a cp-map $\Phi_{t,t+dt}$ the infinitesimal generator L_t has ~~the~~ a general form known as Lindblad equation:

$$\dot{\rho} = -i[H, \rho] + \sum A_k \rho A_k^{\dagger} - \frac{1}{2}\{A_k^{\dagger} A_k, \rho\}$$

Dephasing Dynamics

Single Qubit: $\dot{\rho} = \gamma (Z\rho Z - \rho)$

$\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z) \Rightarrow$

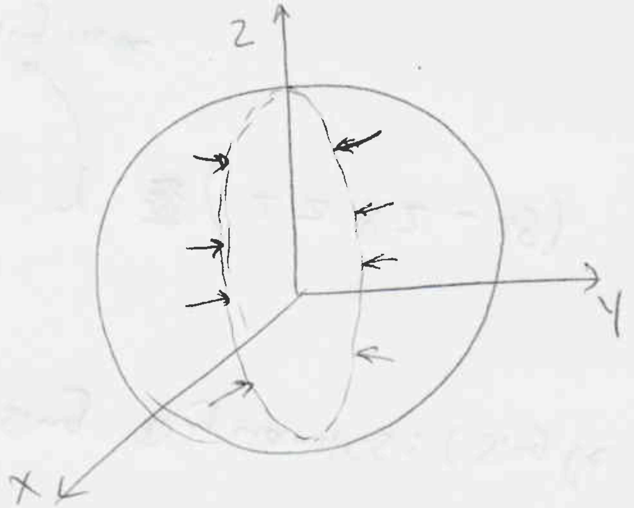
$$\dot{r}_x X + \dot{r}_y Y + \dot{r}_z Z = \gamma \Big(\big(I + r_x(-X) + r_y(-Y) + r_z Z \big) - \big(I + r_x X + r_y Y + r_z Z \big) \Big)$$

$$= -2\gamma (r_x X + r_y Y)$$

$$\Rightarrow \begin{cases} \dot{r}_z = 0 \\ \dot{r}_x = -2\gamma r_x \\ \dot{r}_y = -2\gamma r_y \end{cases} \Rightarrow \rho(t) = \begin{pmatrix} \rho_{00} & \rho_{01} e^{-2\gamma t} \\ \rho_{10} e^{-2\gamma t} & \rho_{11} \end{pmatrix}$$

Dephasing: Loss of Information

(Decoherene process)



$$\dot{\rho} = \gamma\left(2\,\sigma_-\rho\,\sigma_+ - \{\sigma_+\sigma_-,\rho\}\right) \qquad \sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\rho = \frac{1}{2}\left(I + r_x X + r_y Y + r_z Z\right)$$

$$\dot{r}_x X + \dot{r}_y Y + \dot{r}_z Z = \gamma\left(-r_x X - r_y Y + 2(1-r_z)\,Z\right)$$

$$\Longrightarrow \begin{cases} \dot{r}_x = -r_x\gamma \\ \dot{r}_y = -r_y\gamma \\ \dot{r}_z = 2\gamma(1-r_z) \end{cases} \Longrightarrow \begin{cases} r_x = e^{-\gamma t}\, r_x(0) \\ r_y = e^{-\gamma t}\, r_y(0) \\ r_z = 1 + (r_z(0)-1)\,e^{-2\gamma t} \end{cases}$$