Linear Algebra for Quantum Mechanics

- The basic object of linear algebra is a vector space
- A vector space of interest is the set of n-tuples $(x_1, \dots, x_n)$ of complex numbers, $x_i \in \mathbb{C} \ \forall i$.

- A vector is a member of the vector space:

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

- A vector space is equipped with two operations:

  - Addition $\vec{x} + \vec{y} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$

  - Scalar multiplication:

$$z\,\vec{x} = z \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} z x_1 \\ \vdots \\ z x_n \end{pmatrix}$$

- Ket notation of a vector $\vec{x}$: $|x\rangle$

- vector basis : A set of vectors $|v_1\rangle \cdots |v_n\rangle$ such that

$$\forall |x\rangle, \exists c_i \in \mathbb{C}, \quad |x\rangle = \sum c_i |v_i\rangle$$

  - dimension of V
  - Linear independency

- A Linear operator L between two vector spaces V, W:

$$\left.\begin{cases} L v_1 = w_1 \\ L v_2 = w_2 \end{cases}\right\} \Longrightarrow L(c_1 v_1 + c_2 v_2) = c_1 w_1 + c_2 w_2$$

$\forall v_i \in V, \forall w_i \in W, \forall c_i \in \mathbb{C}$.

Basis: $L\left(\sum_i c_i |v_i\rangle\right) = \sum c_i L(|v_i\rangle)$

* Matrix representation of a linear operator:

$$L|v_i\rangle = \sum_j L_{ij} |w_j\rangle$$

Inner product vector space V (Finite dimension Hilbert space)

Some vector spaces are equiped with an inner product:

A function That Takes vectors $|v_1\rangle$ & $|v_2\rangle$ to a complex number. $(|v_1\rangle, |v_2\rangle) = \underbrace{\langle v_1 | v_2 \rangle}$

(Bracket) Quantum mechanic notation

_ Dual vector $\langle v|$ to a vector $|v\rangle$ is a linear operator from V to $\mathbb{C}$: $\langle v|(|v_1\rangle) = \langle v|v_1\rangle = (|v\rangle, |v_1\rangle)$

A function $(\cdot,\cdot) : V \times V \longrightarrow \mathbb{C}$ is an inner product iff

(1) $\left(|v\rangle, \sum_i \alpha_i |w_i\rangle\right) = \sum_i \alpha_i \left(|v\rangle, |w_i\rangle\right)$

(2) $(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$

(3) $(|v\rangle, |v\rangle) \geq 0$ & $(|v\rangle, |v\rangle) = 0$ iff $|v\rangle = 0$.

For $\mathbb{C}^n$: $((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_i x_i^* y_i =$

$(x_1^* \cdots x_n^*)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \langle x|y\rangle \Rightarrow \langle x| = (x_1^* \cdots x_n^*)$

vector norm: $\langle v|v\rangle = \| |v\rangle \|^2$, normalization $\frac{|v\rangle}{\sqrt{\langle v|v\rangle}}$

orthonormal basis: $\{|i\rangle\} \to \langle i|j\rangle = \delta_{ij}$

$$\langle v|w\rangle = \left(\sum_i v_i |i\rangle , \sum w_j |j\rangle \right) = \sum_{ij} v_i^* w_j \delta_{ij}$$

$$= (v_1^* \cdots v_n^*) \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

- Outer product: $\underbrace{(|w\rangle\langle v|)}_{\text{Linear operator}} (|v'\rangle) \equiv |w\rangle \langle v|v'\rangle$

- Completness relation of an orthonormal basis:

$$\left(\sum |i\rangle\langle i| \right) |v\rangle = \sum |i\rangle \langle i|v\rangle = \sum v_i |i\rangle = |v\rangle$$

$$\Rightarrow \sum_i |i\rangle\langle i| = I$$

Matrix representation of L:

$$L = I_w L I_v = \sum \langle w_i | L | v_j \rangle |w_j\rangle\langle v_i|$$

Eigenvalue - Eigenstate of a Linear operator L: (5)

Solve the equation $L|\nu\rangle = \nu|\nu\rangle$.

The scalar ν is called an eigen value.

The vector $|\nu\rangle$ is an eigenstate.

Eigenvalues are solutions of $\underbrace{\det(L - \nu I) = 0}_{\text{characteristic equation.}}$

An operator L is diagonalizable if there exist real number $\{\lambda_i\}$ and vectors $\{|i\rangle\}$ such that $L = \sum \lambda_i |i\rangle\langle i|$

Adjoint and Hermitian operators:

suppose L is a linear operator on a Hilbert space $\mathcal{H}$.

There exist a unique linear operator $A^\dagger$ s.t.

$$(|v\rangle, A|w\rangle) = (A^\dagger|v\rangle, |w\rangle), \quad \langle v|A|w\rangle = \langle A^\dagger v|w\rangle$$

$A^\dagger$: adjoint or Hermitian Conjugate of A.

- Hermition Conjugate of a vector $|v\rangle^{\dagger} \equiv \langle v|$

- Matrix Representation of H.C. : $\left(\sum a_{ij} |i\rangle\langle j|\right)^{\dagger} = \sum a^{*}_{ji} |i\rangle\langle j|$

- Hermitian or self-adjoint : $L = L^{\dagger} \Longrightarrow$ Real eigenvalues (physical)

- normal matrix : $L^{\dagger}L = LL^{\dagger} \longleftrightarrow L$ is diagonalizable

- projector : $P^2 = P \longrightarrow P$ projects to a subspace.
  define

- Unitary : $U^{\dagger}U = I$ and/or $UU^{\dagger} = I$. ~~so what is the~~

  Importance of Unitary maps: $(U|v\rangle, U|w\rangle)$
  $= \langle v|U^{\dagger}U|w\rangle = \langle v|w\rangle$

- positive definite matrix : $(|v\rangle, L|v\rangle) \geq 0$
  $\langle v|L|v\rangle \geq 0$
  $\Downarrow$
  non-negative eigenvalues.

- Trace : $Tr(L) = \sum_i \langle i|L|i\rangle$ $\quad |i\rangle$: orthonormal basis

Tensor product:

A way to merge two vector spaces and form a larger space.

Consider 2 spaces V and W of dimensions m and n.

~~We start with matrix representation a.k.a Kronecker product.~~

The product space $V \otimes W$ is the mn dimension space of tensor product vectors $|v\rangle \otimes |w\rangle \equiv \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$

$$= \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ \vdots \\ v_1 w_m \\ v_2 w_1 \\ v_2 w_2 \\ \vdots \end{pmatrix}$$

If $|i\rangle$ and $|j\rangle$ are orthonormal basis for

V ~~and~~ W then $|i\rangle \otimes |j\rangle$ is the basis for $V \otimes W$

Properties: for scalar z.

1- $z(|v\rangle \otimes |w\rangle) = (z|v\rangle) \otimes |w\rangle = |v\rangle \otimes (z|w\rangle)$

2- $(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$

3- $|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$

Linear operator $A \otimes B$ acting on $V \otimes W$ is defined as

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle$$

$$(A \otimes B) \sum a_i |v_i\rangle \otimes |w_i\rangle = \sum a \, A|v_i\rangle \otimes B|w_i\rangle$$

Matrix Rep., Kronecker product:

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ \vdots & & & \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{pmatrix} \Longrightarrow$$

Inner product:

$$(|\psi_1\rangle \otimes |\psi_2\rangle, |\phi_1\rangle \otimes |\phi_2\rangle) := \langle \psi_1|\phi_1\rangle \langle \psi_2|\phi_2\rangle$$

$\longrightarrow$ orthonormality of bases:

$$(|i_1\rangle \otimes |i_2\rangle, |j_1\rangle \otimes |j_2\rangle) = \langle i_1|j_1\rangle \langle i_2|j_2\rangle = \delta_{i_1 j_1} \delta_{i_2 j_2}$$

A special case of linear operator:

outer product:

$$|\psi_1, \psi_2\rangle\langle \phi_1, \phi_2| = |\psi_1\rangle\langle\phi_1| \otimes |\psi_2\rangle\langle\phi_2|$$

Function of an operator:

spectral decomposition: $A = \sum a|a\rangle\langle a|$

$f(A) = \sum f(a)|a\rangle\langle a|$

$\exp(A) = \sum e^{a}|a\rangle\langle a|$ $\Longrightarrow$ what is the property of the generater of a unitary? and Lie group generator.

Pauli matrices: Z, X, Y

Commutator: $[A,B] = AB - BA$

anti-Commutator: $\{A,B\} = AB + BA$

- Two Hermitian operators A and B are simultaneously diagonalizable if and only if

$[A,B] = 0$ (prove it) for non-degerate A 答

Trace:

Consider an operator A on Hilbert Space $\mathcal{H}$.

$$Tr(A) = \sum_i \langle i | A | i \rangle \qquad \{|i\rangle\}: \text{Complete basis.}$$

Cyclic:

$$Tr(AB) = Tr(BA) \Rightarrow Tr(UAU^\dagger) = Tr(A) \quad \text{For Unitary } U.$$

partial Trace:

Consider an operator A on Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

partial trace of A over $\mathcal{H}_2$ is defined as

$$A = \sum a_{ij\,pq} \, |i\rangle\langle j| \otimes |p\rangle\langle q|$$

where $|i\rangle \in \mathcal{H}_1$ & $|p\rangle \in \mathcal{H}_2$.

partial trace : $Tr_2(A) = \sum a_{ij\,pp} \, |i\rangle\langle j|$.

NORM: (Look at functional analysis books) (11)

A norm $\|\cdot\|$ defines a metric $d(x,y) = \|x-y\|$ on V:

a function that measures the distance. properties:

- $d(x,y) : x, y \longrightarrow \mathbb{R}$
- $d(x,y) = d(y,x)$
- $d(x,x) = 0$
- $d(x,z) \leqslant d(x,y) + d(y,z)$

Cauchy sequence: A sequence of elements x_n of a metric space is called Cauchy if $\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall k, m \geqslant n_0$, $d(x_k, x_m) < \epsilon$.

(Any Cauchy sequence in $\mathbb{C}^n$ has a limit)

Hilbert space:

A Hilbert space is a vector space endowed with an inner product and associated norm & metric, such that any Cauchy sequence in H has a limit in H.

* $\|x\| = \sqrt{\langle x | x \rangle}$

In the case of uncountable infinite dimensional Hilbert space, an orthonormal basis is replaced by a basis $|x\rangle$, as x varies in an appropriate measure set Ω with measure dx, s.t.

$$\langle x | x_1 \rangle = \delta(x - x_1)$$

$$|\psi\rangle = \int_{\Omega} \psi(x) |x\rangle \, dx$$

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space of square integrable functions $L^2(\Omega)$ defined as

$$\forall \psi, \text{ s.t. } \left(\int_{\Omega} \psi^*(x) \psi(x) \, dx \right)^{1/2} < +\infty .$$

is a Hilbert space if ~~equ~~ equipped with the inner product

$$(\psi, \phi) = \int_{\Omega} \psi^*(x) \, \varphi(x) \, dx$$