Home work 7, solutions

problem 1.

To show $G^\dagger G = I$ we need to show that

$$\langle \varphi | G^\dagger G | \varphi \rangle = 1 \quad \forall \; |\varphi\rangle \text{ : state of system and probe}$$

suppose $|\varphi\rangle = \iint \varphi_{xy} |x\rangle |y\rangle \, dx\, dy$

$$\Rightarrow \langle \varphi | G^\dagger G | \varphi \rangle = \iiiint \varphi^*_{x'y'} \varphi_{xy} \langle x' | \langle y' | G^\dagger G |x\rangle |y\rangle \, dx\, dy\, dx'\, dy'$$

$$= \iiiint \varphi^*_{x'y'} \varphi_{xy} \langle x' | x \rangle \langle x'+y' | x+y \rangle \, dx\, dy\, dx'\, dy'$$

$$= \iint |\varphi_{xy}|^2 \, dx\, dy = 1 \checkmark$$

(Normalization property of $|\varphi\rangle$)

problem 2.

system-probe state before measurement

$$G|x\rangle|0\rangle = \int_{-\infty}^{\infty} d\xi \left[\frac{1}{\sqrt{2\pi\Delta^2}} \exp\left(-\xi^2/2\Delta^2\right)\right]^{1/2} G|x\rangle|\xi\rangle$$

$$= \int_{-\infty}^{\infty} d\xi \left[\frac{1}{\sqrt{2\pi\Delta^2}} \exp\left(-\xi^2/2\Delta^2\right)\right]^{1/2} |x\rangle|x+\xi\rangle$$

The measurement operator M_p by definition is

$$M_p|x\rangle = \frac{1}{N}\langle p|G|x\rangle|0\rangle \quad , \; N: \text{normalizer}$$

$$\langle p|G|x\rangle|0\rangle = \int_{-\infty}^{\infty} d\xi \left[\frac{1}{\sqrt{2\pi\Delta^2}} \exp\left(-\frac{\xi^2}{2\Delta^2}\right)\right]^{1/2} |x\rangle \underbrace{\langle p|x+\xi\rangle}_{e^{-ip(\xi+x)}}$$

$$= \left(\frac{2\Delta^2}{\pi}\right)^{1/4} \exp(p^2\Delta^2)\exp(-ipx)|x\rangle$$

$$\Longrightarrow M_p = \left(\frac{2\Delta^2}{\pi}\right)^{1/4} \exp(p^2\Delta^2)\exp(-ipX)$$