Homework 6, solutions

problem 1. As we learned before we need to find the Lie-algebra generated by $\{Z_1 \otimes Z_2, X_1+X_2, Y_1+Y_2, Z_1+Z_2\}$ via calculating the commutator of the generators.

step 1.

$$[Z_1 Z_2, X_1+X_2] = +2i(Y_1 Z_2 + Z_1 Y_2) \quad \text{(new)} \ ✓$$

$$[Z_1 Z_2, Y_1+Y_2] = -2i(X_1 Z_2 + Z_1 X_2) \quad \text{(new)} \ ✓$$

$$[Z_1 Z_2, Z_1+Z_2] = 0 \quad \text{(nothing)} \ X$$

$$[X_1+X_2, Y_1+Y_2] = [X_1, Y_1] + [X_2, Y_2] = i(Z_1+Z_2) \quad X$$

similarly

$$[X_1+X_2, Z_1+Z_2] \quad X$$

$$[Y_1+Y_2, Z_1+Z_2] \quad X$$

step 2.

$[Z_1Z_2, Y_1Z_2+Z_1Y_2] = -2i(X_1Z_2+Z_1X_2)$ X

$[Z_1Z_2, X_1Z_2+Z_1X_2] =$ X

$[X_1+X_2, Y_1Z_2+Z_1Y_2] = 4i(Z_1Z_2-Y_1Y_2)$ ✓

$[Y_1+Y_2, Y_1Z_2+Z_1Y_2] = 2i(Y_1X_2+X_1Y_2)$ ✓

$[Z_1+Z_2, Y_1Z_2+Z_1Y_2] = 2i(-X_1Z_2-Z_1X_2)$ X

$[X_1+X_2, X_1Z_2+Z_1X_2] = 2i(-Y_1X_2-X_1Y_2)$ X

$[Y_1+Y_2, X_1Z_2+Z_1X_2] = 4i(-Z_1Z_2+X_1X_2)$ ✓

$[Z_1+Z_2, X_1Z_2+Z_1X_2] = 2i(Y_1Z_2+Z_1Y_2)$ X

step 3. If you continue on the commutations for the new elements $(Z_1Z_2-Y_1Y_2, X_1Z_2+Z_1X_2)$ you will not find any new generator

Therefore the full algebra is generated by

$$\{Z_1Z_2,\ X_1+X_2,\ Y_1+Y_2,\ Z_1+Z_2,\ Y_1Z_2+Z_1Y_2,\ X_1Z_2+Z_1X_2,$$
$$Z_1Z_2-Y_1Y_2,\ Y_1X_2+X_1Y_2,\ X_1X_2-Z_1Z_2\}.$$

The Lie-algebra dimension is 9 as opposed to 15 the dimension of the su(4) algebra

$\Longrightarrow$ Not completely controllable.

The controllability of the system

$$H=\gamma\omega_0 Z_1Z_2+\gamma B_x(t)(X_1+X_2)+\gamma B_z(t)(Z_1+Z_2)$$

is similar to the previous case since

$$[X_1+X_2,\ Z_1+Z_2]\propto Y_1+Y_2$$

However this second system is slower as it takes some time to generate the generator Y_1+Y_2.

problem 2.

I) A basis for $su(n)$ is simply

$$\{ i(|k\rangle\langle l| + |l\rangle\langle k|),\ |k\rangle\langle l| - |l\rangle\langle k| \},\quad l,k \in \{1 \dots n\}$$

A basis for $sp(\frac{n}{2})$ can be obtained by considering the generic matrix form of it

$$\begin{pmatrix} L_1 & L_2 \\ -L_2^* & L_1^* \end{pmatrix} \text{ with } \begin{cases} L_1^\dagger = -L_1 \\ L_2 = L_2^T \end{cases}$$

A basis:

$$\Big\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (|k\rangle\langle l| - |l\rangle\langle k|),\ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \otimes (|k\rangle\langle l| + |l\rangle\langle k|),$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes (|k\rangle\langle l| + |l\rangle\langle k|),\ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \otimes (|k\rangle\langle l| + |l\rangle\langle k|) \Big\}$$

for $k, l \in \{1 \dots \frac{n}{2}\}$