Homework 6, solutions

problem 1. As we learned before we need to find the Lie-algebra generated by
\[ \{ Z_1 \otimes Z_2, X_1 + X_2, Y_1 + Y_2, Z_1 + Z_2 \} \]
via calculating the commutator of the generators.

Step 1.

\[ [Z_1 Z_2, X_1 + X_2] = 2i(Y_1 Z_2 + Z_1 Y_2) \quad \text{(new)} \quad \checkmark \]
\[ [Z_1 Z_2, Y_1 + Y_2] = -2i(X_1 Z_2 + Z_1 X_2) \quad \text{(new)} \quad \checkmark \]
\[ [Z_1 Z_2, Z_1 + Z_2] = 0 \quad \text{(nothing)} \quad \times \]
\[ [X_1 + X_2, Y_1 + Y_2] = \{X_1, Y_1\} + \{X_2, Y_2\} = i(Z_1 + Z_2) \quad \times \]

Similarly

\[ [X_1 + X_2, Z_1 + Z_2] \quad \times \]
\[ [Y_1 + Y_2, Z_1 + Z_2] \quad \times \]
Step 2.

\[
\begin{align*}
\{ z_1 z_2, Y_1 z_2 + z_1 Y_2 \} &= -2i (X_1 z_2 + z_1 X_2) X \\
\{ z_1 z_2, x_1 z_2 + z_1 x_2 \} &= X \\
\{ x_1 + x_2, Y_1 z_2 + z_1 Y_2 \} &= 4i (2_i z_2 + x_1 Y_2) X \\
\{ y_1 + y_2, Y_1 z_2 + z_1 Y_2 \} &= -2i (Y_1 X_2 + x_1 Y_2) X \\
\{ x_1 + x_2, Y_1 z_2 + z_1 Y_2 \} &= 2i (-x_1 z_2 - z_1 x_2) X \\
\{ y_1 + y_2, x_1 z_2 + z_1 x_2 \} &= 2i (Y_1 z_2 + z_1 Y_2) X \\
\{ X_1 + X_2, x_1 z_2 + z_1 x_2 \} &= 2i (-Y_1 X_2 - x_1 Y_2) X \\
\{ Y_1 + Y_2, x_1 z_2 + z_1 x_2 \} &= 4i (-z_1 z_2 + x_1 x_2) X \\
\{ z_1 + z_2, x_1 z_2 + z_1 x_2 \} &= 2i (Y_1 z_2 + z_1 Y_2) X \\
\{ z_1 + z_2, Y_1 Y_2, x_1 z_2 + z_1 x_2 \} &= 2i (Y_1 z_2 + z_1 Y_2) X
\end{align*}
\]

Step 3. If you continue the commutations for the new elements \((z_1 z_2 - y_1 y_2, x_1 z_2 + z_1 x_2)\) you will not find any new generator.
Therefore the full algebra is generated by

\[
\begin{align*}
&Z_1 Z_2, X_1 + X_2, Y_1 + Y_2, Z_1 + Z_2, Y_1 Z_2 + Z_1 Y_2, X_1 Z_2 + Z_1 X_2, \\
&Z_1 Z_2 - Y_1 Y_2, X_1 X_2 + X_1 Y_2, X_1 X_2 - Z_1 Z_2
\end{align*}
\]

The Lie-algebra dimension is 9 as opposed to 15, the dimension of the su(4) algebra.

\[\implies \text{Not completely controllable.} \]

The controllability of the system \( H \)

\[H = \psi w_0 Z_1 Z_2 + \psi B_x(t) (X_1 + X_2) + \psi B_y(t) (Z_1 + Z_2)\]

is similar to the previous case since \( \mathfrak{h} \)

\[ [X_1 + X_2, Z_1 + Z_2] \propto Y_1 + Y_2 \]

However this second system is slower as it takes some time to generate the generator \( Y_1 + Y_2 \).
Problem 2.

I) A basis for $\text{su}(n)$ is simply

$$\left\{ i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \quad l, k \in \{1, \ldots, n\}$$

A basis for $\text{sp}(n/2)$ can be obtained by considering the generic matrix form of it

$$\begin{pmatrix} L_1 & L_2 \\ -L_2^* & L_1^* \end{pmatrix}$$

with \( L_1^T = -L_1 \) and \( L_2 = L_2^T \).

A basis:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

for \( k, l \in \{1, \ldots, n/2\} \).