Homework 4, Solutions

1)

1-1) The easiest solution that comes to my mind is to turn on the B_x component of the magnetic field for a time ~~past~~ $T = \frac{1}{gB_x} \cdot \frac{\pi}{2}$ in order to generate a unitary X. This is easy to see by the relation $e^{i\vec{a}\cdot\vec{\sigma}} = I\cos(a) + i(\vec{n}\cdot\vec{\sigma})\sin(a)$ that you learned in HW 1.

Similarly you can just turn B_z for $T = \frac{1}{gB_z} \cdot \frac{\pi}{2}$ to generate Z.

1-2) From the relations $T = \frac{\pi}{2} \cdot \frac{1}{gB_x (\text{or } B_z)}$ it is clear that a ~~stronger~~ stronger fiedel B_x (B_z) result a shorter time dynamics.

In general a system with a stronger Hamiltonian (more energy) has faster dynamics.

Therefor if we assume unbonded contral of the magnetic field that means the generation of a unitary in principal can take infinitely small time.

Problem 2) The total system-bath unitary at time $T=4\tau^+$ is

$$U_{SB}(T=4\tau^+) = Z F X F Z F X F$$

where F is the unitary evolution of the system and bath in a time internal τ, $F = e^{-i\tau H_{SB}}$

Using Hint 1 : $U_{SB}(T=4\tau^+) = -\underbrace{ZFZ}_{=e^{-i\tau Z H_{SB} Z}}\underbrace{YFY}_{=e^{-i\tau Y H_{SB} Y}}\underbrace{XFX}_{=e^{-i\tau X H_{SB} X}}F$

$$\begin{cases} X H_{SB} X = X B_X - Y B_Y - Z B_Z \\ Y H_{SB} Y = -X B_X + Y B_Y - Z B_Z \\ Z H_{SB} Z = -X B_X - Y B_Y + Z B_Z \\ H_{SB} = X B_X + Y B_Y + Z B_Z \end{cases} \Rightarrow X H_{SB} X + Y H_{SB} Y + Z H_{SB} Z + H_{SB} = 0 \quad (*)$$

From the second hint we know that the product

$$e^{-i\tau Z H_{SB} Z}\, e^{-i\tau Y H_{SB} Y}\, e^{-i\tau X H_{SB} X}\, e^{-i\tau H_{SB}}$$ is

equal to $e^{-i\tau(Z H_{SB} Z + Y H_{SB} Y + X H_{SB} X + H_{SB}) + O(\tau^2)}$

We showed above that the relation $(*)$ is zero

Therefore $H_{SB}(T=4\tau) = e^{O(\tau^2)}$. (Q.E.D.)

In the case that $H_{SB} = X B_X + Y B_Y + Z B_Z$, the bath operator B_0 is zero. In more general case that the bath has some internal Hamiltonian B_0 then to the first order of τ $U_{SB}(T) = e^{-i I \otimes B_0}$, still no action on the system.