1) Zero potential $V(x) = 0$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2}$$

Looking for special solution $\psi(x, t) = f(t) \Phi(x)$

$$i\hbar f(t) \frac{d \Phi}{dx} = \frac{\hbar^2}{2m} f(t) \frac{d^2 \Phi}{dx^2}$$

$$\Rightarrow \frac{f(t)}{f(t)} = \frac{i\hbar}{2m} \frac{d^2 \Phi}{dx^2}$$

Assume this ratio is equal to $i\omega$.

$$\frac{f(t)}{f(t)} = i\omega \Rightarrow f(t) = A e^{-i\omega t}$$

for some constant $A$.

$$\frac{i\hbar}{2m} \frac{d^2 \Phi}{dx^2} = -i\omega \frac{d^2 \Phi}{dx^2} = -\frac{\omega(2m)}{\hbar} \Phi,$$

$$\Rightarrow \Phi(x) = B e^{ikx} \text{ or } Be^{-ikx}$$
Therefore an special solution will be,
\[ \Psi(x) = C e^{i(kx - \omega t)} + C e^{-i(kx - \omega t)} \]
with \[ k = \frac{\alpha \omega}{\hbar} \]

Therefore the general solution will be a superposition of these special solutions.
\[ \Psi(x, t) = \int c_+(\omega) e^{i(kx + \omega t)} \, dw + \int c_-(\omega) e^{-i(kx + \omega t)} \, dw \]

Normalization Condition: \[ \int |\Psi(x, t)|^2 \, dx = 1 \]
\[ \Rightarrow \int \, dw \left( |c_+(\omega)|^2 + |c_-(\omega)|^2 \right) = 1 \]

For the case of a mono chromatic wave function, \( C_+ \) and \( C_- \) are delta functions which is not a real physical function.
2) State of the system and probe after time $t$ is

$$
S_{sp}(t) = e^{-iH_{sp}t} S_{sp}(0) e^{iH_{sp}t}
$$

1- The probability of the $k$'th outcome is

$$
\text{prob}(k) = \text{Tr} \left( I_S \otimes P_k S_{sp}(t) \right) = \text{Tr} \left( P_k \text{Tr}_S (S_{sp}(t)) \right)
$$

System identity operator

2- The collapsed state of the system & probe is

$$
S_{sp}^{(k)} = \frac{(I \otimes P_k) S_{sp}(t) (I \otimes P_k)}{\text{Tr}(I_S \otimes P_k S_{sp}(t))}
$$

$$
= \frac{(I \otimes P_k) e^{-iH_{sp}t} S_{sp}(0) e^{iH_{sp}t} (I \otimes P_k)}{\text{prob}(k)}
$$