Solutions to HW #1

1. We solve $\det(M-\lambda I)=0$ for the eigenvalue λ.

$$\det\begin{pmatrix}1-\lambda & 1 \\ 0 & 1-\lambda\end{pmatrix}=0 \longrightarrow \boxed{\lambda=1} \quad \text{(degenerate eigenvalue)}$$

Eigen-vector: $\begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix}=0 \Longrightarrow y=0$

$$\Longrightarrow |\lambda_x\rangle = \begin{pmatrix}x \\ 0\end{pmatrix}$$

$$\Longrightarrow \underline{\nexists\, x_k}, \text{ s.t. } M=\sum_k |\lambda_{x_k}\rangle\langle\lambda_{x_k}| .$$

(does not exist x_k)

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2. A positive Matrix M satisfies $\langle\psi|A|\psi\rangle \geq 0$ for any vector $|\psi\rangle$.

choose $|\psi\rangle = \begin{pmatrix}0 \\ 0 \\ a_1 \\ \vdots \\ a_s \\ 0 \\ 0\end{pmatrix}$: all elements with indices out side S is zero & take some arbitrary value inside S.

For this special selection of $|\psi\rangle$,

$$\langle\psi|M|\psi\rangle = \langle\bar{\psi}|M(S)|\bar{\psi}\rangle$$ where $M(S)$ is the principal submatrix as defined in the HW.

$|\bar{\psi}\rangle$ is the reduced vector obtained by stacking the elements $|\bar{\psi}\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_s \end{pmatrix}$

Since the vector $|\bar{\psi}\rangle$ is arbitrary then

$\forall |\bar{\psi}\rangle, \langle\bar{\psi}|M(s)|\bar{\psi}\rangle \geq 0 \Longrightarrow M$ is positive.

3. Let start from calculating $(\vec{a}\cdot\vec{\sigma})^2$

$$(\vec{a}\cdot\vec{\sigma})^2 = \sum_{\substack{i,j\in\\ \{x,y,z\}}} a_i a_j \sigma_i \sigma_j = \sum_{ij} a_i a_j \left(\delta_{ij} I + i\sum_k \epsilon_{ijk}\sigma_k\right)$$

$$= \sum_i a_i^2 = a^2$$

$$\Longrightarrow (\vec{a}\cdot\vec{\sigma})^{2n} = a^{2n}$$

$$\Rightarrow (\vec{a}\cdot\vec{\sigma})^{2n+1} = a^2\, \vec{a}\cdot\vec{\sigma}\,.$$

Using these two identities we can find

$$e^{i\vec{a}\cdot\vec{\sigma}} = \sum_n \frac{1}{n!}(i\vec{a}\cdot\vec{\sigma})^n = \sum_{\text{even } n} \frac{(-1)^{n/2}}{n!} a^n + \sum_{\text{odd } n} \frac{(-1)^{\frac{n-1}{2}}}{n!} a^{n-1} (i\vec{a}\cdot\vec{\sigma})$$

$$= \cos(a)\, I + i\sin(a)\, (\hat{a}\cdot\vec{\sigma})$$

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(4) It is easy to see that $\Phi(\rho) \geqslant 0$ for any matrix ρ.

$$\text{If } \rho = \sum_\lambda \lambda |\lambda\rangle\langle\lambda| \Rightarrow \Phi(\rho) = \sum_{\alpha,\lambda} \lambda\, E_\alpha |\lambda\rangle\langle\lambda| E_\alpha^\dagger$$

$$\forall |\psi\rangle,\ \langle\psi|\Phi(\rho)|\psi\rangle = \sum \lambda\, |\langle\psi|E_\alpha|\lambda\rangle|^2 \geqslant 0 \quad \checkmark$$

$$\sum |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|) = I\otimes\Phi(|\psi\rangle\langle\psi|)$$

where $|\psi\rangle = \sum_i |i\rangle$

$$\Longrightarrow \quad I \otimes \Phi(|+\rangle\langle+|) = \Phi'(|+\rangle\langle+|)$$

where the map Φ' is CP with operators $\{I \otimes E_\alpha$

~~therefor~~ therefor $\quad \Phi'(\rho) = I \otimes \Phi(\rho) \geqslant 0$

for any ρ in the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}$.