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Quantum feedback control: how to use verification theorems and viscosity solutions to find optimal protocols

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While feedback control has many applications in quantum systems, finding optimal control protocols for this task is generally challenging. So-called ‘verification theorems’ and ‘viscosity solutions’ provide two useful tools for this purpose: together they give a simple method to check whether any given protocol is optimal, and provide a numerical method for finding optimal protocols. While treatments of verification theorems usually use sophisticated mathematical language, this is not necessary. In this article we give a simple introduction to feedback control in quantum systems, and then describe verification theorems and viscosity solutions in simple language. We also illustrate their use with a concrete example of current interest.

Keywords: quantum feedback control; verification theorems; viscosity solutions

1. Introduction

In quantum feedback control (QFC), an observer continuously monitors a quantum system [1,2], and uses the information from this measurement, as it is obtained, to control the system by continually modifying the Hamiltonian and/or the observable being measured. Over the last decade the topic of QFC has generated an increasing amount of theoretical [3–40] and experimental [41–44] work. This is due partly to experimental progress in micro- and mesoscopic quantum systems [41–49], partly because QFC possesses a wide range of potential applications [5,6,50–74], and partly because of its connection with fundamental questions in quantum mechanics [11,35,74,75].

Unlike in classical feedback control, in QFC the measurement that is part of the feedback loop affects the dynamics of the system [1,2]. Even so, QFC is actually contained within the general framework of classical control theory. This is because, even if the measurement in the feedback process is continually modified as the system evolves, quantum systems are merely specific examples of noisy non-linear dynamical systems. As a result, the techniques of classical control theory are usually applicable to quantum systems. However, most of the powerful results of control theory apply only to linear systems, and are therefore not relevant to the majority of quantum feedback control problems. An important exception are the ‘verification theorems’ [76–80]. These are applicable to feedback control in any dynamical system, and have two uses. The first is to test whether or not a given feedback protocol is the \textit{optimal} protocol for a given task. If one has devised a protocol, by intuitive means or otherwise, it is very useful to be able to check if it is optimal, especially if the protocol admits an analytic solution. Knowing that a protocol is optimal is useful, but so is knowing that it is not – this tells us that there may yet be hidden and unexpected things to learn about the given problem. The second use of verification theorems is that they provide a systematic numerical procedure for finding optimal feedback protocols.

The literature on verification theorems is not easily accessible to most quantum physicists, however, because it is written in the jargon of axiomatic probability theory (filtrations, adapted processes and the like). Further, the most widely applicable verification theorem, proved in the last few years, requires the use of ‘superderivatives’ and ‘viscosity solutions’ [81], a subject unfamiliar to most theoretical physicists. Our purpose here is to explain, in a straightforward manner and without technical jargon, how to check if a feedback protocol is optimal, and how to find optimal protocols numerically. We also give a concrete example of the former, involving an optimal feedback protocol that we reported recently in [82], and for which the viscosity verification theorem is essential. We note that we were first introduced to verification theorems by the article of Wiseman and Bouten, in
which they used these theorems to prove the optimality of a number of quantum rapid-purification protocols [51,73]. Since the usefulness of verification theorems in QFC is clear, we felt that an accessible article on this topic would be useful.

We begin in the following section by introducing the subject of quantum feedback control itself. While the concepts are simple, it does require stochastic calculus, and this is unfamiliar to many physicists. We thus start by discussing this aspect of QFC a little. We then present the equations of motion for a continuously monitored system, and show how the effect of feedback can be easily included. We next show how to quantify the problem of optimising the feedback protocol for a given task. Having done this we now know what the optimisation problem is for any given control task, and we can show how to use the ‘classic’ verification theorem to check whether a feedback protocol is optimal. This is the subject of Sections 3 and 4. The classic verification theorem is sufficient so long as the evolution of the system does not satisfy this condition, however. An example is a protocol which switches abruptly at some time from one Hamiltonian to another. In this case the evolution has continuous derivatives everywhere except at the point(s) where the protocol switches. To check optimality in this more general case we require a more general verification theorem. This theorem, while very similar to the first, requires the use of viscosity solutions. In Section 5 we explain how to calculate a superderivative (and subderivative), and how to show if a function is a ‘viscosity solution’ of a given differential equation. We can then show how to use the more general verification theorem to determine the optimality of a feedback protocol that has discontinuities, and we do this in Section 6. With this we have achieved our main goal. In Section 8 we show how the verification theorems also provide a numerical method to find optimal feedback protocols, and in Section 9 we show how, with almost no modification, the same techniques work for time-optimal control problems. Section 10 concludes with a brief summary.

2. Quantum feedback control

2.1. Including feedback in quantum mechanics

The dynamics of an isolated quantum system is given by Schrödinger’s equation. If we describe the system using a density matrix, \( \rho \), then this equation is

\[
\dot{\rho} = -\frac{i}{\hbar}[H, \rho].
\]

(1)

If there is an environment that introduces noise into the system, then one can often include the effects of this environment by adding the terms of a Lindblad master equation to the above Hamiltonian evolution.\(^1\) The equation of motion for \( \rho \) becomes

\[
\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \gamma \mathcal{D}[c] \rho,
\]

where we have defined

\[
\mathcal{D}[c] \rho = 2c^\dagger \rho c - c^\dagger c \rho - \rho c^\dagger c.
\]

(3)

Here \( \gamma \) is a rate constant, and \( c \) is an operator that depends on the coupling of the system to the environment. To add to the equation of motion the effect of monitoring the system, we simply add another set of terms. Before we do so, however, let us explain exactly what we mean by monitoring.

Monitoring a system means that we obtain a continuous readout of some property of the system. Let us say that we are monitoring the position of an object. This means that in each tiny time-step \( dt \), we obtain a little bit of information about this position, \( x \). Why a tiny bit? Why not the precise position? To understand this, first recall that all real measurements (classical as well as quantum) have some inaccuracy, which means that the measurement result is \( x \) plus some random number (the error). This random number limits the amount of information we have about \( x \) (that is, it limits the accuracy to which we can pinpoint \( x \) after obtaining the result). Now, no physical measurement process can extract a finite amount of information in an infinitely short time, because the speed of any interaction is necessarily finite. Thus, in a time interval \( dt \), the amount of information must also be infinitesimal, which means that the error must tend to infinity as \( dt \to 0 \). For a classical measurement the measurement result in a time interval \( dt \) is therefore given by

\[
R = x + \epsilon,
\]

(4)

where \( x \) is the true value of \( x \), and \( \epsilon \) must have a bigger variance the smaller \( dt \). It turns out that the quantum equivalent of this is \([1]\)

\[
R = \langle X \rangle + \epsilon,
\]

(5)

where \( X \) is the operator for the observable \( x \).\(^2\) It turns out that if the error is Gaussian (being by far the most common measurement error because of the central limit theorem, and the one we will consider here), the variance of \( \epsilon \) must be proportional to \( 1/dt \) (and thus the error is proportional to \( 1/(dt)^{1/2} \)). For further details on the reason for this, we refer you to \([1,83]\).

Because of this the value of \( R \) becomes, strictly speaking, infinite as \( dt \to \infty \). So instead of writing our equations in terms of \( R \), we write them instead in
terms of $\text{R}dt$, which we will denote by $\text{d}r$. This is a
perfectly finite quantity. So we have

$$\text{d}r = \langle X \rangle \text{d}t + g \text{d}W, \quad (6)$$

where $g$ is an arbitrary constant determining the
amount of the noise, and $\text{d}W$ is a Gaussian random
variable with mean zero and variance equal to $\text{d}r$.
Thus, in each interval $\text{d}t$, $\text{d}W$ is picked from the
probability density

$$P(\text{d}W) = \frac{1}{(2\pi \text{d}t)^{1/2}} \exp \left[ -\frac{\text{d}W^2}{2(\text{d}t)} \right]. \quad (7)$$

Now that we know precisely what we mean by a
continuous measurement of a physical quantity $x$; we
mean a measurement that provides a continuous
stream of measurement results $\text{d}r(t)$. Now we need to
know how the state $\rho$ changes in each time interval as
a result of the measurement. In the interval $\text{d}t$, the state
$\rho(t)$ goes to $\rho(t) + \text{d}\rho$, where $[1]$

$$\text{d}\rho = -kD[X]\rho + (2k)^{1/2}(X\rho + \rho X - 2\langle X \rangle\rho) \text{d}W, \quad (8)$$

where $k = 1/(8g^2)$ and as usual $\langle X \rangle = \text{Tr}[X\rho]$. Loosely
speaking, $k$ gives the rate at which the measurement extracts
information about $x$. Here the random increment $\text{d}W$ is precisely the same increment that appears in the measurement record $\text{d}r$. This makes sense: the change in the state of the system in the time interval $\text{d}t$ depends upon the measurement result in that time interval. As a result we can also write the
above equation for $\text{d}\rho$ directly in terms of the
measurement record:

$$\text{d}\rho = -kD[X]\rho + 4k(X\rho + \rho X - 2\langle X \rangle\rho)
\times (\text{d}r - \langle X \rangle \text{d}t). \quad (9)$$

Normally one would divide $\text{d}\rho$ by $\text{d}t$ to get the time
derivative $\dot{\rho}$, and write the equation of motion for $\rho$
using this derivative. However, $\text{d}W/\text{d}t$ is rather
pathological (just as $r$ is above), so when writing differential
equations involving $\text{d}W$ we keep them in terms of differentials. Thus, the full equation of motion for a
noisy system that is also continuously monitored is

$$\text{d}\rho = -(i/\hbar)[H, \rho] \text{d}t + \mathcal{D}[c]\rho \text{d}t - kD[X]\rho \text{d}t
+ (2k)^{1/2}(X\rho + \rho X - 2\langle X \rangle\rho) \text{d}W. \quad (10)$$

This equation is referred to as a stochastic equation,
because it contains a term that fluctuates randomly.

To be able to manipulate stochastic equations, one
needs to use the rules of stochastic calculus. This boils
down entirely to the rule $\text{d}W^2 = \text{d}t$. On first sight this
is a very strange relation, because $\text{d}W$ is random, and $\text{d}t$
is deterministic. However, note that $(\text{d}W^2) = \text{d}t$.
Because of this, if one divides any tiny time interval into
many smaller sub-intervals, and adds up the many
increments of $\text{d}W^2$, the result is a deterministic amount in the limit in which there are infinitely many sub-
intervals $[84]$. The result is that in the continuum limit it
is always true that $\text{d}W^2 = \text{d}t$. For further details
regarding the manipulation and solution of stochastic
equations we refer you to $[1, 2, 83, 84]$.

Now that we have the evolution equation for a
monitored system, we want to include feedback in the
dynamics. This is very simple. All we do is specify that
the observer can change the Hamiltonian of the
system, $H$, and also if we wish, the measured
observable $X$, and allow $H$ and $X$ to be some function
of the measurement results. Specifically, $H$ and $X$ at
time $t$ are allowed to be any function of the measurement
results, $\text{d}r(t)$, obtained up until that time. We can write $H$, for example, as $H(\mathcal{F}(t))$, where $\mathcal{F}(t) = \int_0^t f(t, t') \text{d}r(t')$ where $f(t, t')$ is an arbitrary function of its arguments. Because the density matrix at time $t$ completely determines the future behaviour of the system (given the system Hamiltonian), all optimal feedback protocols can be obtained by making $H(t)$ and $X(t)$ functions of $\rho(t)$ (and possibly the current time, $t$). The observer obtains $\rho(t)$ simply by using the
measurement results to solve Equation (10) from the
initial time up until time $t$. Thus, we choose $H =
H(t, \rho)$ and $X = X(t, \rho)$. The equation for the evolution
of $\rho$, including feedback, is then simply

$$\text{d}\rho = -(i/\hbar)[H(t, \rho), \rho] \text{d}t - kD[X(t, \rho)]\rho \text{d}t
+ (2k)^{1/2}(X(t, \rho)\rho + \rho X(t, \rho) - 2\langle X(t, \rho) \rangle\rho) \text{d}W
+ \mathcal{D}[c]\rho \text{d}t. \quad (11)$$

The problem of feedback control is to determine $H(t, \rho)$
and/or $X(t, \rho)$ in order to achieve a desired evolution as
closely as possible in the presence of noise.

### 2.2 The goal of control

In feedback control the goal is usually one of three things:
to arrive at a desired state at a given time $T$; to
produce an evolution that is as close as possible to
some desired evolution; or to reach a given state as
quickly as possible. The first two are called ‘finite-
horizon’ control, and the third ‘time-optimal’ control.
We will consider the first two of these when introdu-
cing the verification procedures. These procedures can
be applied equally well to the third, with minor
modification, and we discuss this in Section 9.

The reason that the first two control objectives fall
in the same category is that the first is merely a special
case of the second: in the former we require the system to be close to a specific pure state at a given 'horizon time' \( T \); in the latter we require that the system be as close as possible to some given (time dependent) pure state, \( |\psi(t)\rangle \), at every time. This desired state is called the target state, or the target evolution. (Note that there is little point in choosing the target state to be mixed – mixing merely represents a lack of knowledge regarding the state, and this is not beneficial in a control setting.)

Now consider the second control objective. To define precisely how well a control protocol achieves the objective, we need to choose a measure of the distance from one quantum state to another to quantify what we mean by the system being close to the target state. There are a number of measures we can use, such as the fidelity \([85]\), the distinguishability \([86]\), or the quantum relative entropy \([87]\), to name but three. If we choose the fidelity then we want our control protocol to minimise

\[
F(t) = 1 - \langle \psi(t)|\rho(t)|\psi(t)\rangle. \tag{12}
\]

Since this is a function of time, to obtain a single quantity to minimise we must combine \( F(t) \) at all the different times into a single number. To use verification theorems, and indeed the majority of results of optimal control theory, we must combine the \( F \)'s in a simple fashion: the function to minimise must be merely a weighted sum of the \( F \)'s at different times. Since time is continuous, this means that the function to minimise is \( J = \int_0^T \mathcal{L}(t)F(t)\,dt \), for some \( \mathcal{L}(t) \) and final time \( T \). This means that the function to minimise is simply an integral over time of a function of the state \( \rho \) at each time. Fortunately this form is quite reasonable and well-motivated, and is rather general. For example, we can weight different times differently in the integral if some are more important than others. If the final time is especially important, we can give it extra weight by including it by itself in addition to the integral (that is, place a delta function weighting at time \( T \)). We can also include more general functions of the state, since we can place any function of the form \( \mathcal{L}(\rho(t),t) \) in the integral. For example, if we just want to control the expectation value of a particular observable then \( \mathcal{L} \) can be a function of this expectation value. Or we could choose \( \mathcal{L} \) to be the von Neumann entropy of the state, as another example.

There is one more crucial thing to add. If we could both extract information from the system infinitely fast, and apply infinite forces to the system (control Hamiltonians that induce infinitely fast dynamics) then we could easily make the system follow any target evolution exactly, even in the presence of environmental disturbance and other noise. Feedback control is only non-trivial if the measurement rate and/or feedback forces are constrained, which they always are in real applications. There are two ways to place constraints on the control inputs and the measurement rate. One is simply to place a fixed upper bound on them, and attempt to minimise \( J \) under these constraints. The alternative is to include a function of the measurement rate and feedback Hamiltonian in \( L \). This means that in minimising \( J \) we are trying to find the protocol that gives us the best evolution, while at the same time keeping the control forces as small as possible. The size of the feedback forces, and the rate at which we extract information, are referred to as the 'costs' of the control protocol. Because one often includes these in the functional to minimise, \( J \), this functional is usually called the cost.

Finally, since the system is driven by noise, the actual cost (effort) expended by the controller on a given run will depend on the specific values that the noise takes for that run. These specific values are called the noise realisation. Since the actual cost will vary from run-to-run, it is sensible to have the control protocol minimise the average value of \( J \), where the average is over all noise realisations.

To summarise succinctly the above discussion, a feedback control problem is defined by the equation of motion of the system we are trying to control, and a cost of the form

\[
J = \left\langle \int_0^T \mathcal{L}(\rho(t),t,H_{fb}(t),X(t))\,dt + M(\rho(T)) \right\rangle_n, \tag{13}
\]

that we choose based on our control objective, and that we want to minimise. Here \( H_{fb} \) is the part of the Hamiltonian that gives the forces applied by the controller (the 'control Hamiltonian'), and we have separated out the contribution to the cost of the final state, calling this \( M(\rho(T)) \). It is useful to do this because in many control problems it is, in fact, only the final state that is important, so that one sets \( L = 0 \). The angle brackets with the subscript 'n', \( \langle \ldots \rangle_n \), indicate that the value of the integral is averaged over all possible noise realisations.

3. The classic verification theorem

We now explain how to use the 'classic' verification theorem of optimal control theory to determine whether a given protocol is the optimal one.

3.1. First, a few definitions...

We will now move to a new notation. This is to make the presentation easier, and to employ the same
notation as used by most control theory texts. We write the general dynamical equation for the system we are trying to control as

\[ \dot{x} = A(t, x, u(x, t)) \, dt + B(t, x, u(x, t)) \, dW. \]  

(14)

Here the state of the system is given by the vector \( x \), and the control inputs to the system are denoted by the vector \( u \) (which is usually a function of the current state, \( x \)).\(^3\) Thus, for quantum feedback control, the vector \( x \) is the vector of the elements of \( \rho \):

\[ x \leftrightarrow \rho. \]  

(15)

The vector \( u \) is the set of all parameters that we can vary. If we write the control Hamiltonian as

\[ H_{\text{fb}}(t) = \mu_1(t)H_1 + \mu_2(t)H_2 + \cdots + \mu_i(t)H_i + \cdots, \]

(16)

where the \( \mu_i \) are real numbers, then \( u \) is the set of these \( \mu_i \), along with a set of numbers that determine the observable we choose to measure, \( X(t) \). If we write \( X \) as

\[ X = UD U^\dagger, \]

(17)

where \( U \) is unitary and \( D \) is diagonal, then \( X \) is determined in the most general case by the diagonal elements of \( D \), and the \( N(N+1)/2 \) angles \( \theta_k \) that parametrise an \( N \)-dimensional unitary [88]. Thus,

\[ u \leftrightarrow \{ \mu_i, D, \theta_k \}. \]  

(18)

The vector \( A \) is any vector-valued function of \( x, u \) and \( t \), and gives the deterministic dynamics of the system. It is determined by the Hamiltonian \( H \) and the noise operator \( c \). (Yes – the noise on the system actually gives deterministic motion, because we are considering the evolution of the density matrix. It is only the measurement that makes the evolution stochastic.) The matrix-valued function \( B \) gives the stochastic part of the evolution. It is determined by the measured observable \( X(t) \). The vector \( dW \) is a vector of mutually independent Wiener noises. As usual we will take the initial time to be \( t = 0 \), denote the initial state of the system as \( x(0) = x_0 \), and we will call the final (horizon) time \( t = T \).

The function \( u(t, x(t), t) \) completely defines the control protocol. It tells us what control inputs (what control Hamiltonian) to apply at any time \( t \), for every possible state of the system at that time. Thus, \( u(x(t), t) \) is the control protocol.

The control inputs, \( u \), are usually subject to limits. In general these limits can be defined by any closed region in the vector space in which \( u \) lives. The simplest set of conditions would be

\[ u_{\text{low}} \leq u(x, t) \leq u_{\text{high}}. \]  

(19)

Note that the region (and thus \( u_{\text{low}} \) and \( u_{\text{high}} \)) is not a function of \( t \) or \( x \); there is only one region that bounds the values of control inputs for the duration of the control. The procedure we will describe is general enough, however, to handle a limiting region that varies with time.

As described above, the control objective is to minimise a cost, \( J \). Using our new notation, this cost is

\[ J = \left\langle \int_0^T L(x, u(x,s), s) \, ds + M(x(T)) \right\rangle_n, \]

(20)

where \( L \) and \( M \) are functions that we choose. For the verification procedure, we have to introduce one more quantity, called the cost function, \( C(x,t) \). This is defined as the value of \( J \) over the interval \([t,T]\), given that the system is at state \( x \) at time \( t \):

\[ C(x, t) = \left\langle \int_t^T L \, ds + M(x(T)) \right\rangle_n. \]  

(21)

Note that \( C(x,t) \) is the average cost that will have to be paid over the time remaining, given we have reached time \( t \). For this reason it is often called the ‘cost-to-go’.

### 3.2. The verification procedure

To determine whether a given control protocol, \( u = f(x,t) \) is optimal, one performs the following four steps:

1. Integrate the equations of motion of the system to calculate the cost function, \( C(x(t), t) \), for this protocol.
2. Check that \( C \) satisfies two continuity conditions. These are that
   \[ \frac{\partial C}{\partial t} \quad \text{and} \quad \frac{\partial^2 C}{\partial x^2} \]
   are continuous. Here \( \partial^2 C/\partial x^2 \) denotes the matrix of second derivatives of \( C \).
3. Determine whether or not \( C(x,s) \) satisfies the following differential equation (called the Hamilton–Jacobi–Bellman (HJB) equation):
   \[ \frac{\partial C}{\partial t} = \max_v \left[ G \left( x, v, \frac{\partial C}{\partial x}, \frac{\partial^2 C}{\partial x^2} \right) \right], \]
   with the final condition \( C(x,T) = M(x(T)) \).\(^4\)
We will give the function $G$ below. Before we do so, we note that the maximum is taken over the allowed values of a vector $v$. The allowed values of $v$ are those of $u$: $v$ can only take values in the region that bounds the values of $u(x,t)$. Note that because $G$ is a function of $t$ and $x$, one must maximise $G$ separately at each time and at each value of $x$. Because of this, the value of $v$ that maximises $G$ will in general be different at different values of $t$ and $x$. Thus, the $v$ that maximises $G$ is in general a function of $t$ and $x$: $v_{\text{max}}(x,t)$.

Now we come to the function $G$. This is

$$ G = \frac{1}{2} \text{Tr} \left[ B(t, x, v) \frac{\partial^2 C}{\partial x^2} B(t, x, v) \right] - A \cdot \frac{\partial C}{\partial x} - L(t, x, v). \quad (24) $$

Note that $A$ and $B$ are, respectively, the vector and matrix that appear in the equation of motion for the system, $\partial C/\partial x$ is the vector of first derivatives of $C$, and $\partial^2 C/\partial x^2$ is the matrix of second derivatives. To check that $C$ satisfies the differential equation given by Equation (23), the only potentially tricky thing is determining what the maximum of $G$ is for each $t$ and $x$—however, this can be fairly easy, depending on the system.

(4) Check that the $u(x,t)$ for your protocol maximises $G$ (if this maximum is unique, then this also means that $u(x,t) = v_{\text{max}}(x,t)$).

4. Using the verification theorem: an example

We now apply the verification theorem to a concrete example. This example was solved in [82], but without the details of the analysis. The problem involves feedback control of a three-state quantum system (otherwise known as a qutrit). The purpose is to prepare the system in a given pure target state at time $T$, but it is only the measurement strength $k$ that is bounded; it is assumed the feedback Hamiltonian available to the controller generates evolution that is much faster than the measurement strength, and the noise in the system. It is also assumed that the controller has the ability to implement any Hamiltonian for the qutrit.

Because of the strong feedback Hamiltonian, the problem becomes one of maximising the largest eigenvalue of the density matrix. The reason for this is that, at any desired time, the controller can quickly evolve the system (that is, apply a unitary operator) so that the eigenvector corresponding to the largest eigenvalue is equal to the target state. For a given density matrix, this unitary operation maximises the probability that the system is in the target state, and is thus the optimal application of the Hamiltonian part of the feedback. In this case the probability that the system is in the target state is equal to the largest eigenvalue of the density matrix. The feedback control problem in this case is therefore to choose the measured observable, $X(t)$, as the system evolves, so as to maximise the largest eigenvalue of the density matrix at time $T$.

Since the feedback Hamiltonian is very large, it can also be used to continually rotate the eigenvectors of the density matrix so the density matrix eigenbasis remains constant with time. The result of this is that, when we express $X$ in terms of this eigenbasis, the equations of motion for the eigenvalues of the density matrix are decoupled from those of the eigenvectors, and so we only need to consider the motion of the eigenvalues to find the optimal control protocol.

In addition to the restriction on the measurement strength, we restrict the observable $X$ to having equi-spaced eigenvalues. Since the measurement part of the master equation is invariant under the transformation $X \rightarrow zX$, where $z$ is a real number, $X$ can be taken to be traceless without loss of generality. As a result, the measured observable is of the form

$$ X = U \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} U^\dagger, \quad (25) $$

where $U$ is any unitary.

Before solving the problem, we specialise to the ‘regime of good control’. This is the regime in which the probability that the system is in the target state, $P$, is close to unity [35,82]. This means that $P = 1 - \Delta$, where $\Delta \ll 1$. One can then simplify the dynamics of the density matrix by expanding to first order in $\Delta$. Recall that the goal of the feedback in this case is to optimise the largest value of the density matrix, and this problem is completely defined by the dynamics of the eigenvalues alone.

We now denote the largest eigenvalue by $\lambda_0$, the second largest by $\lambda_1$, and the smallest by $\lambda_2$. Since $\sum \lambda_i = 1$, we have only two independent dynamical variables, $\lambda_1$ and $\lambda_2$. In the regime of good control, under a measurement of $X$, the equations of motion are [82]

$$ d\lambda_1 = -8 \left[ |X_{10}|^2 \lambda_1 - |X_{21}|^2 \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \right] dt + 8^{1/2} (X_{00} - X_{11}) \lambda_1 dW, \quad (26) $$

$$ d\lambda_2 = -8 \left[ |X_{20}|^2 \lambda_2 + |X_{12}|^2 \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \right] dt + 8^{1/2} (X_{00} - X_{22}) \lambda_2 dW, \quad (27) $$
where the matrix elements of $X$ are those in the eigenbasis of the density matrix at the current time. Note that the task is to choose $X(t)$ so as to maximise $\lambda_0(T)$, which means minimising $\Delta(T) = \lambda_1(T) + \lambda_2(T)$.

Our candidate for the optimal control protocol, suggested in [82], is to choose $X$ at each time so that, in the eigenbasis of the density matrix, it is

$$X = \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (28)$$

This is suggested as the optimal protocol only up until the point at which $\lambda_1$ has been reduced to the value of $\lambda_2$. We now use the procedure described in the previous section to determine whether or not this protocol is optimal. (If you really want to burn this procedure into your brain, you can stop reading now and try it.)

To begin we need to obtain the cost function. For this we need the time evolution of the thing we want to minimise (the cost), $\Delta$, under the suggested protocol. Under the protocol, the only non-zero elements of $X$ are $X_{01} = X_{10} = a$, and the equations of motion become

$$\frac{d\lambda_1}{dt} = -8a^2\lambda_1, \quad \frac{d\lambda_2}{dt} = 0. \quad (29)$$

These are easy to solve, with the result that $\Delta(t) = \lambda_1(0)\exp(-8a^2t) + \lambda_2(0).$ Recall that the cost function, $C(\lambda_1, \lambda_2, t)$, is the value of $\Delta$ at the final time, $T$, given that we started with the values $\lambda_1$ and $\lambda_2$ at time $t$. So this is

$$C(\lambda_1, \lambda_2, t) = \lambda_1\exp[-8a^2(T-t)] + \lambda_2. \quad (31)$$

Note that all the derivatives of $C$ are continuous, so the continuity conditions for the verification theorem are satisfied.

Now we have to calculate the function $G$. The ingredients for this are the equations of motion, Equations (26) and (27), and the first and second derivatives of $C$ with respect to $\lambda_1$ and $\lambda_2$. Calculating these derivatives we find that

$$G(X) = 8\left[\lambda_1|X_{10}|^2\exp[-8a^2(T-t)] + \lambda_2|X_{20}|^2\right] + 8\left(\frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2}\right)|X_{21}|^2 \times (1 - \exp[-8a^2(T-t)]). \quad (32)$$

We must now find the maximum of $G(X)$ over all $X$, under the constraint Equation (25). Thus, we must maximise $G$ over all unitaries $U$. It is important to note that the $a$ that appears in $G(X)$ above is not part of the optimisation; this has already been fixed by the protocol. To perform the maximisation, we note first that we are only considering times $t$ such that $\lambda_1 \geq \lambda_2\exp[8a^2t].$ Because of this, we can show that $G$ will be maximised if we maximise

$$F(X) = 8\lambda_2\left[\eta|X_{10}|^2 + |X_{20}|^2 + \zeta|X_{21}|^2\right]. \quad (33)$$

where $\eta$ and $\zeta$ are constants satisfying $\eta > 1 > \zeta.$ (Specifically $\eta = \lambda_1\exp[8a^2(T-t)]/\lambda_2,$ and $\zeta = \eta [\exp(8a^2(T-t)] - 1]/(\lambda_1/\lambda_2 - 1).$) We do this maximisation over $U$ numerically using Matlab’s fminsearch function. (This function uses the Nelder–Mead direct search algorithm.) This shows that the maximum is obtained when $X$ is given by Equation (28). Substituting this into $G(X)$ we have

$$\max_U G(X) = \lambda_18a^2\exp[-8a^2(T-t)]. \quad (34)$$

Calculating the time derivative of $C$ we have

$$\frac{\partial C}{\partial t} = \lambda_18a^2\exp[-8a^2(T-t)]. \quad (35)$$

Equations (34) and (35) are equal. Thus, the cost function satisfies the Hamilton–Jacobi–Bellman equation (Equation (23)), and so the protocol is optimal.

Recall that we have only tested the protocol up until the point at which the lowest two eigenvalues equalise. If $t$ is the starting time, and $T$ the final time, then this means that

$$T \leq t + \frac{\ln(\lambda_1/\lambda_2)}{8a^2}. \quad (36)$$

5. Viscosity solutions

The purpose of ‘viscosity solutions’ (a branch of the theory of differential equations) is to allow one to define a continuous function to be a solution of a second order differential equation, even though the function does not have well-defined derivatives. The reason that this subject is useful to us here, is that optimal feedback protocols often produce cost functions ($C(\lambda_1, \lambda_2, T)$ in the previous section) that are not differentiable at some (usually small, finite) set of points. It turns out that such cost functions are viscosity solutions of the HJB equation, and as a result one can check their optimality by using a verification theorem that is very similar to the classic verification theorem in the previous section. So it turns out yet again that an initially rather odd-seeming branch of mathematics has a direct application.

To use the more general verification theorem we first have to know how to show if something is
a viscosity solution to a given DE. To do this we need two new definitions, that of a superderivative (also called a superjet) and a subderivative (or subjet). To begin with, note that if a function \( f(y) \) is twice-differentiable at point \( y_0 \), then

\[
f(y) - f(y_0) = p \cdot (y - y_0) + \frac{(y - y_0)^T Q (y - y_0)}{2}
\]

(37)
to second order in \( y - y_0 \), where

\[
p = \frac{\partial f}{\partial y}
\]

(38)
is the vector of first derivatives, and \( Q \) is the matrix of second derivatives:

\[
Q_{ij} = \frac{\partial^2 f}{\partial y_i \partial y_j}, \quad \text{or} \quad Q = \frac{\partial^2 f}{\partial y^2}.
\]

(39)

We now define the superderivative, \( J^+(y_0) \) of a function at the point \( y_0 \) as the set of all vectors \( p \) and matrices \( Q \) such that

\[
f(y) - f(y_0) \leq p \cdot (y - y_0) + \frac{(y - y_0)^T Q (y - y_0)}{2}
\]

(40)
to second order in \( y - y_0 \). The subderivative, \( J^-(y_0) \) is similarly defined by replacing the \( \leq \) with a \( \geq \). In feedback control, the functions one deals with are usually twice-differentiable everywhere except at a small number of points. So as an example, let us calculate the super- and subderivatives of the function

\[
f(y) = f_-(y) = \ln y, \quad 0 < y \leq 1,
\]

(41)

\[
f(y) = f_+(y) = (y - 1)^2, \quad y \geq 1,
\]

(42)
at the point \( y = 1 \) (where it is not differentiable). To do this we first expand \( f_\pm \) about the point \( y = 1 \), using the first two terms of the Taylor series. This gives

\[
f_-(y) - f_-(1) = \Delta y - (\Delta y)^2,
\]

(43)

\[
f_+(y) - f_+(1) = 2\Delta y^2,
\]

(44)
to second order in \( \Delta y = y - 1 \). So for the real numbers \( p \) and \( Q \) to be in the superderivative set, \( J^+(1) \), we need them to satisfy the following two equations simultaneously:

\[
\Delta y - (\Delta y)^2 \leq p\Delta y + Q(\Delta y)^2, \quad \Delta y < 0,
\]

(45)

\[
2\Delta y^2 \leq p\Delta y + Q(\Delta y)^2, \quad \Delta y > 0.
\]

(46)

Note that the second order terms are irrelevant unless the first order terms on each side are equal. Examining equation Equation (46) we see that the condition is satisfied so long as \( p > 0 \), or \( p = 0 \) and \( Q \geq 2 \). Examining Equation (45), the condition is satisfied when \( p \leq 1 \) (remember that \( \Delta y \) is negative for Equation (45)), or \( p = 1 \) and \( Q \geq -1 \). Since (45) and (46) must be satisfied simultaneously, the superderivative is the set

\[
J^+(1) = \begin{cases} 
0 < p < 1, & Q \in (-\infty, \infty), \\
p = 0, & Q \geq 2, \\
p = 1, & Q \geq -1.
\end{cases}
\]

(47)

For the real numbers \( p \) and \( Q \) to be in the subderivative, \( J^-(1) \), they must satisfy

\[
\Delta y - (\Delta y)^2 \geq p\Delta y + Q(\Delta y)^2, \quad \Delta y < 0,
\]

(48)

\[
2(\Delta y)^2 \geq p\Delta y + Q(\Delta y)^2, \quad \Delta y > 0.
\]

(49)

From Equation (48) we find that \( p \geq 1 \), and from Equation (49) that \( p \leq 0 \). Since these cannot both be true at once, \( J^-(1) \) is the empty set:

\[
J^-(1) = \emptyset.
\]

(50)

Now that we know what superderivatives and subderivatives are, we can define a viscosity solution to a differential equation. We first write our differential equation in the form

\[
F\left(y, f, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y^2}\right) = 0.
\]

(51)

A continuous function \( f(y) \) is a viscosity solution of this PDE if it is true that

\[
F(y, f(y), p, Q) \geq 0, \quad \text{whenever} \quad (p, Q) \in J^+(y),
\]

\[
F(y, f(y), p, Q) \leq 0, \quad \text{whenever} \quad (p, Q) \in J^-(y).
\]

If the function \( f \) is a viscosity solution of a PDE at a given point, and its derivatives exist at this point, then it is a solution in the normal sense. Therefore, to determine if a function is a viscosity solution to a PDE, one first checks that it is a solution to the PDE in all regions where its first two derivatives are defined. Then one calculates the super- and subderivatives of \( f \) at those points where its derivatives are not defined, and checks that \( f \) is a viscosity solution at those points.

6. The enhanced verification theorem

The more general verification theorem, due to Zhou and collaborators [79,80], states that \( u(x,t) \) is an
optimal feedback protocol so long as the following conditions are satisfied by the cost function, $C(x(t),t)$:

1. The cost function must be a viscosity solution of the HJB equation, and it must be true that $C(x(T),T) = M(T)$. To determine whether these conditions are satisfied, one first writes the HJB equation in the form

$$-\frac{\partial C}{\partial t} + \max_{u} \left[ G \left( x, v, \frac{\partial C}{\partial x}, \frac{\partial^2 C}{\partial x^2} \right) \right] = 0.$$  

If we set the vector $y$ in Equation (51) to $y = (t, x_1, \ldots, x_N)$, then the above HJB equation is exactly of the form given in Equation (51). Because the HJB equation does not depend on any second derivatives that include $t$, to determine whether $C$ is a viscosity solution, we do not need to consider these second derivatives when calculating the super- and subderivatives. Thus, for the purposes of solving the HJB equation, the superderivative at $x_0$ and time $t_0$, $J^+(x_0, t_0)$, is the set of values of $(q, p, Q)$ such that

$$C(x, t) \leq C(x_0, t_0) + q\Delta t + p \cdot \Delta x + \Delta x^T Q \Delta x,$$

where $\Delta t = t - t_0$ and $\Delta x = x - x_0$. Naturally the subderivative is defined in the same way, with the inequality reversed. Armed with these super- and subderivatives one uses the procedure described in the previous section to determine if $C$ is a viscosity solution to the HJB equation.

2. Check that $u(t)$ maximises $G$.

3. For each $t$ there must be a $q(t)$, $p(t)$ and $Q(t)$, such that:

$$(q(t), p(t), Q(t)) \in J^+(x(t), t)$$

and

$$q(t) = G(x(t), u(x(t), t), p(t), Q(t)).$$

where $x(t)$ is the evolution of the system under our feedback protocol $u(x, t)$. The reason that we no longer need the maximisation in this equation, is because $C$ is a viscosity solution to the HJB equation (Equation (52)), and $u(t) = u(x(t), t)$ maximises $G$. This means that setting $x = \hat{x}(t)$ and $v = u(\hat{x}(t), t)$ already maximises $G$.

7. Using the enhanced verification theorem: an example

For this example we consider the same control problem as the previous example (Section 4), but we consider the problem for all evolution times, not merely up until the two smallest eigenvalues have equalised. From this point onwards, the protocol suggested in [82] involves rapidly switching the measured observable between $X$, given by Equation (28), and the observable

$$X_2 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}. \quad (56)$$

This is equivalent to measuring $X$ while rapidly switching the eigenvectors of the eigenvalues $\lambda_1$ and $\lambda_2$. In the limit of fast switching, the equations of motion of both eigenvalues are identical, and given by

$$\dot{\lambda}_i = -4a^2\lambda_i, \quad i = 1, 2. \quad (57)$$

Starting at time $t$, and evolving the eigenvalues using the original protocol up until time $t + \tau$, where $\tau = \ln(\lambda_1/\lambda_2)/(8a^2)$, and then using the second protocol from that time until time $T$, we find that, for $T > t + \tau$, the cost function is

$$C^- (\lambda_1, \lambda_2, t) = 2(\lambda_1\lambda_2)^{1/2} \exp \left[ -4a^2(T - t) \right]. \quad (58)$$

For times $T < t + \tau$, the cost function is still given by Equation (31), and we will call this $C^+$. The cost function is continuous everywhere, but is no longer differentiable at $t = T - \tau$. Thus, we cannot use the classic verification theorem on the protocol, but must use the enhanced verification theorem. This requires that we show that $C$ is a solution to the HJB equation on the intervals in which it has continuous derivatives, and that it is a viscosity solution at the point $t = T - \tau$. We already know that it satisfies the HJB equation on the interval $t \in [T - \tau, T]$ (that is, when the evolution time is less than $\tau$). So we examine next the interval $t \in [0, T - \tau]$. To calculate $G$ we need the first and second derivatives of $C^-$ with respect to $\lambda_1$ and $\lambda_2$. These are

$$\frac{\partial C^-}{\partial \lambda_j} = \frac{C^-}{2\lambda_j}, \quad (59)$$

$$\frac{\partial^2 C^-}{\partial \lambda_j^2} = -\frac{C^-}{4\lambda_j^2}, \quad (60)$$

$$\frac{\partial^2 C^-}{\partial \lambda_1 \partial \lambda_2} = -\frac{C^-}{4\lambda_1 \lambda_2}. \quad (61)$$

The expression we get for $G$ this time is

$$G(X) = \left( X_{11} - X_{22} \right)^2 C^- + 4 \left( |X_{10}|^2 + |X_{20}|^2 + |X_{21}|^2 \right) C^- . \quad (62)$$
We therefore need to find the $U$ that maximises
\[ F(X) = (X_{11} - X_{22})^2 + 4 \left( |X_{10}|^2 + |X_{20}|^2 + |X_{21}|^2 \right). \]

Once again we perform this optimisation numerically using Matlab’s fminsearch. It turns out that there are many unitaries that maximise $F(X)$, and the unitaries that give the $X$’s defined by Equations (28) and (56) both do so. So we have
\[ \max_x G(X) = 4a^2 C^- . \] (63)

This is exactly the time derivative of $C^-$, so the cost function does satisfy the HJB equation on the interval $[T - \tau, T]$.

Our final task is to determine whether the cost function is a viscosity solution to the HJB equation at $t = T - \tau$. To do this we would usually proceed to calculate the super- and subderivatives, which would involve expanding $C^+$ and $C^-$ in their Taylor series as describe in Section 5. However, it turns out that in this case we can take a shortcut. First we note that the protocol involves switching between two measurement operators, so the cost function must be a solution to the HJB equations for both. The two HJB equations are
\[
\frac{\partial C}{\partial t} = 8|X_{10}|^2 \frac{\partial C}{\partial a_1}, \quad (64)
\]
\[
\frac{\partial C}{\partial t} = 8|X_{20}|^2 \frac{\partial C}{\partial a_2}. \quad (65)
\]

Because these do not contain the second derivatives, we do not need to calculate the $Q$ part of the super- and subderivatives to determine if $C$ is a viscosity solution. We only need the first order parts, $q$ and $p$. But since the first derivatives of $C$ are continuous at $t = T - \tau$, $q$ and $p$ are exactly these derivatives.\(^6\) So all we need to do is to substitute $C^+$ (or $C^-$) into Equations (64) and (65), and check that it is a solution to both. Indeed it is, and thus $C$ is a viscosity solution of the HJB equation, and we can conclude that the suggested protocol is optimal.

8. Finding optimal protocols

The two verification theorems we have described above also provide a method to search for optimal protocols. To see how this works we return to the Hamilton–Jacobi–Bellman equation, which is
\[
\frac{\partial C}{\partial t} = \max_v \left[ G \left( x, v, \frac{\partial C}{\partial x}, \frac{\partial^2 C}{\partial x^2} \right) \right]. \quad (66)
\]

Recall that $C(x,t)$ is the total average cost over the remaining time interval, $[t, T]$, given that the system has state $x$ at time $t$. The elements of the vector $v$, in general being functions of the state $x$ and time $t$, are the parameters that give the control protocol. The evolution of the system, $x(t)$, and the cost function $C(x,t)$, are determined by the choice of $v$. If the pair $v$ and $C$ solve the HJB equation when substituted into it, then $v$ is an optimal protocol.

The fact that allows us to use the HJB equation to search for optimal protocols, is that every pair of $v$ and $C$ satisfies the equation
\[
\frac{\partial C}{\partial t} = G \left( x, v, \frac{\partial C}{\partial x}, \frac{\partial^2 C}{\partial x^2} \right). \quad (67)
\]

However, $G$ will only be maximised if $v$ is an optimal protocol. This means that we can find an optimal protocol by searching over the allowed space of controls $v$ for the $v$ that maximises $G$. Note that if $v$ is time-dependent, then we will have to search over all functions $v$ for one that maximises $G$ at every time $t$. This can certainly be a challenging task – nevertheless, it does provide a systematic procedure. Further, the problem simplifies in the following way: one can perform the search by starting close to the final time, and stepping backwards. That is, solving for the optimal $v$ in the small time interval $[T - \Delta t, T]$, for all states $x$ at the start of this interval. Then, with $v(x,t)$ for that interval fixed at the optimal result obtained, solving for the interval $[T - 2 \Delta t, T - \Delta t]$, and so-on moving backwards. Because of the form of the cost, Equation (13), this backwards-in-time procedure will find a globally optimal $v$ for the whole interval. (This point is discussed in most control texts that include the Bellman equation – see e.g. [89,90].)

If $v$ is not a function of time, but only of the state, $x$, then the task is significantly easier, since not only is the search space reduced, but it usually means that if $G$ is maximised at a single time, it is maximised at all times, so we need only evaluate $G$ at a single time. If one can obtain an analytic solution to the equations of motion for every $v$, then one can obtain an analytic expression for $G$ in terms of $v$. In this case numerical minimisation is likely to be easy.

9. An alternative form for the cost: time-optimal control

So far we have taken the cost to be a function of the dynamical variables, integrated over time, including a separate contribution at the final time. There is another useful form for the cost function that allows us to use exactly the same verification procedures. This alternative cost is what we use if we want to minimise the
time taken to reach a particular event. This kind of control problem is called *time-optimal control*. In this case we define a function $h(x(t), t)$ of the dynamical variables (and perhaps of time), and the goal is to minimise the time taken for $h$ to cross a fixed value $h_c$, called the *threshold*. The cost is then defined as the expectation value of the time remaining before $h$ crosses the threshold. Everything else is the same as before, the only change is this form of the cost. As before, the cost function, $C(x, t)$, is the average value of the cost from time $t$ until we finish, given that the state at time $t$ is $x$. That means that $C(x, t)$ is the average value of time it will take to cross the threshold, given that the current time is $t$ and current state is $x$. Thus, if we define $x_c$ as the state (or set of states) for which $h = h_c$, then $C(x_c, t) = 0$.

The only change in the verification theorem is that the HJB equation changes a little. It is now [76]

$$\frac{\partial C}{\partial t} = \max_x \left[ G(x, v, \frac{\partial C}{\partial x}, \frac{\partial^2 C}{\partial x^2}) \right] - 1, \quad (68)$$

where $G$ is the same as before, except that it no longer contains the function $L$:

$$G = -\frac{1}{2} \text{Tr} \left[ B^T(t, x, v) \frac{\partial^2 C}{\partial x^2} B(t, x, v) \right] - A \cdot \frac{\partial C}{\partial x}. \quad (69)$$

Note that the HJB equation does not contain either the function $h(x(t), t)$, or the threshold value $h_c$. These have already played their role by determining $C$.

10. Conclusion

We have shown how to use verification theorems to determine whether a given feedback protocol is optimal. The procedure is quite straightforward, and involves: (1) solving the equations of motion to obtain the cost function, (2) checking two simple continuity conditions on the cost function, (3) checking that the cost function is a solution of the HJB equation, and (4) checking that the protocol maximises the RHS of the HJB equation.

If the cost function does not satisfy the continuity conditions (step 2 above), then there is an enhanced verification procedure that can be used. This is essentially the same as the previous procedure, but one checks instead that the cost function is a *viscosity* solution of the HJB equation.

We have also described how the HJB equation can be used, at least in principle, to find optimal protocols. For non-trivial problems this will usually, but not always, require a numerical implementation, and if so is likely to be numerically intensive. What we have not yet done is to show how the HJB equation is derived. This is actually quite straightforward, and we give this derivation in the Appendix.

Notes

1. We can do this so long as the environment is memoryless (Markovian).
2. While it makes sense that for a classical measurement of $x$ the result should be the true value plus an error, it is not so clear why the expression $(x + \epsilon)$ also makes sense. Nevertheless, it turns out that the two forms are completely equivalent. It follows that the only way that a measurement of $x$ can determine the expectation value of $x$ precisely is to determine the true value of $x$ precisely, at which point the expectation value and the true value are the same.
3. So long as the control at time $t$ does not depend explicitly on the state of the system at earlier times, the control is referred to as Markovian. In fact, so long as the current state is the observers complete state-of-knowledge determined from the measurement record, in theory there is never any need to have the current control depend upon the system at earlier times, since the complete future evolution is entirely determined by the current state.
4. The terminal condition on the HJB equation, $C(x, T) = M(x(T))$, is always satisfied by the cost function, since this condition follows immediately from its definition.
5. We note that in their proof of the enhanced verification theorem, the authors of [80] took the equation of motion for the system to be real, while the quantum SME is complex. This should not cause any problem, because a complex vector differential equation can always be written as a real vector differential equation with twice the number of variables. However, to be sure we checked with F. Gozzi, who confirmed that the proof applies to complex variables without change.
6. Strictly speaking this is only true if the super and subderivatives are nonempty, but if they are empty then $C$ is automatically a viscosity solution to the HJB equation at $t = T - \tau$.
7. The terminal condition on the time-optimal HJB equation is $C(x_c, T) = 0$. This is, of course, satisfied by the cost function, $C$, since it is part of its definition.

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Appendix 1. The Hamilton–Jacobi–Bellman equation

To begin with we define the value function, \( V(x,t) \), as the minimum of the cost function over all possible control protocols:

\[
V(x(t), t) = \min_u C_u(x(t), t),
\]

(70)

where we have added the subscript \( u \) to the cost function to make explicit the fact that it depends on the choice of the control, \( u(x(t), t) \). Recall that the cost function, \( C_u(x(t), t) \), is the average value of the cost that will be incurred from time \( t \) until the final time. (The cost function is defined in Section 3.)

The starting point for the HJB equation is the fact that the value function satisfies the recursion relation

\[
V(x(t), t) = \min_u \left( \int_t^T L ds + V(x(t'), t') \right),
\]

(71)

where \( L = L(u(x(t), t), x(t), t) \) is the instantaneous cost incurred at time \( t \). This recursion relation follows immediately from two facts. The first is that any protocol that gives the optimal control from time 0 to \( T \), must also give the optimal control from any time \( t' > 0 \) to \( T \). Recall that the optimal protocol tells us what control to apply at every time \( t \), for every state \( x(t) \) that the system could be in at that time.

The second fact is that the total cost is merely a sum (integral) over the separate costs \( L(t) \) at each time \( t \). Because of this the total average cost incurred from time \( t \) onwards is the sum of the cost incurred from time \( t \) to \( t' \), and the cost incurred from \( t' \) onwards. That is

\[
C_u(x(t), t) = \left( \int_t^{t'} L ds + C_u(x(t'), t') \right).
\]

(72)

Adding to this the fact that the optimal protocol \( u \) minimises both \( C_u(x(t), t) \) and \( C_u(x(t'), t') \), we obtain the recursion relation for \( V \) (Equation (72)).

The HJB equation can be extracted from Equation (72) by setting \( t' = t + dt \) and applying Ito’s rule, \( dW^2 = dt \). Note that this means we must keep all the differentials up to the second order. To begin with we have

\[
V(x(t), t) = \min_u \left( \int_t^{t+dt} L ds \right) \times \langle V(x(t + dt), t + dt) \rangle_u.
\]

(73)

We now substitute into this the Taylor expansion for \( V(x + dx, t + dt) \), being

\[
V(x + dx, t + dt) = V(t, x) + \frac{\partial V}{\partial t} + dx \frac{\partial V}{\partial x} + dx^2 \frac{\partial^2 V}{\partial x^2} \cdot dx,
\]

and the equation of motion for the system (Equation (14)), being

\[
x + dx = x + A(t, x, u(x, t)) dt + B(t, x, u(x, t)) dW.
\]

(74)

The result is the HJB equation

\[
\frac{\partial V}{\partial t} = \max_u \left\{ \frac{1}{2} \text{Tr} \left[ B(t, x, u)^T \frac{\partial^2 C}{\partial x^2} B(t, x, u) \right] - \frac{\partial C}{\partial x} - L(t, x, u) \right\}.
\]

(75)

Notice that in the last step we used the fact that \( -\min(f) = \max(-f) \) for any function \( f \). A rigorous derivation of the HJB equation can be found in many stochastic control textbooks, an example being [76].