EE290O
Homework 5: Differential Geometry

Problem 1: (Submanifold) Let M be a manifold and $V \subset M$ an open subset. Prove that if $\{\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^n\}$ is an atlas on M then $\{\phi_\alpha|_{V \cap U_\alpha}\}$ is an atlas on V. Conclude that V is a manifold. This defines a submanifold.

Problem 2: (Lie-Algebra Basis) There is an algorithmic way to generate a basis for the Lie-Algebra of a given set of matrices $\{A_1, ..., A_k\}$. The recursive procedure is as follows:

Step 0: List the vectors of depth 0, $\{A_1, ..., A_k\}$.

Step k:

1. Calculate the Lie brackets of the elements of depth $k-1$ obtained at step $k-1$, with the elements of depth 0. This way one obtains elements of depth k.
2. Out of the elements obtained, select the ones that form a linearly independent set along with the ones obtained up to step $k-1$.
3. Stop the procedure if there is no new vector or if the dimension of the linearly independent set is n^2-1 or n^2.

The set of matrices obtained this way forms a basis of the dynamical Lie algebra .

Reference: Domenico D'Alessandro, Introduction to Quantum Control and Dynamics, Chapman & Hall (2008).

Now consider a two-qubit system with the Hamiltonian $\omega_0 Z_1 \otimes Z_2$. Suppose we can control these qubits by magnetic fields with X component only:

Case I: (Global control with a single tunable field $B(t)$)

$$H = \omega_0 Z_1 \otimes Z_2 + \gamma B(t)(X_1 \otimes I_2 + I_1 \otimes X_2) \tag{1}$$

Case II: (Local control with double tunable fields $B_1(t), B_2(t)$)

$$H = \omega_0 Z_1 \otimes Z_2 + \gamma B_1(t)(X_1 \otimes I_2) + B_2(t)(I_1 \otimes X_2) \quad (2)$$

Question: For both cases use the above algorithm to construct a basis for the Lie-algebra generated by the internal and control Hamiltonians. What is the dimension of the Lie-algebra for each case?

Case I: $\{Z_1 \otimes Z_2, X_1 \otimes I_2 + I_1 \otimes X_2\}$
Case II: $\{Z_1 \otimes Z_2, X_1 \otimes I_2, I_1 \otimes X_2\}$