EE290O
Homework 4: Dynamical Decoupling

Problem 1: (Pulse Generation) We want to apply a unitary transformation on a single spin $S = \frac{\hbar}{2}(X\hat{x} + Y\hat{y} + Z\hat{z})$. We can do this by turning on some magnetic field $B = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$. As we learned in the class, the dynamics of the spin is determined by the Hamiltonian $H = gS.B$ for some constant g. Suppose we can arbitrarily control the components of the magnetic field $\{B_x(t), B_y(t), B_z(t)\}$.

1-1) Give simple controls of functions $\{B_x(t), B_y(t), B_z(t)\}$ in-order to generate the unitary X and the unitary Z.

1-2) Argue that the time it takes to generate the unitary X or Z becomes shorter as a stronger magnetic field is applied.

Problem 2: (Dynamical Decoupling) We learned in the class that the dynamics of an open quantum system is not unitary due to interactions with the environment. In this problem we learn about an open loop control technique for cancellation of the environmental noise. In this so-called dynamical decoupling technique a series of strong fast pulses are applied on the system which *on average* neutralizes the system-bath coupling, say decouple system from bath (environment). Here is the general idea:

1- Suppose system is coupled to the bath via the Hamiltonian H_{SB}. The unitary evolution of the system and bath from time $t = 0$ to $t = \tau$ would be

$$U_{SB}(\tau) = e^{-i\tau H_{SB}}. \quad (1)$$

2- At time $t = \tau$ a strong pulse on "system only" would generate a unitary transformation P on the system. Notice that since we assume the pulse is very strong this unitary rotation happens instantaneously (Problem 1). Therefore the total unitary evolution of the system and bath is

$$U_{SB}(\tau^+) = Pe^{-i\tau H_{SB}}. \quad (2)$$

3- Let the system and bath evolves for another time period τ yielding the unitary

$$U_{SB}(2\tau) = e^{-i\tau H_{SB}} P e^{-i\tau H_{SB}}. \tag{3}$$

4- followed by another strong pulse that generates the unitary $P^\dagger$

$$U_{SB}(2\tau^+) = P^\dagger e^{-i\tau H_{SB}} P e^{-i\tau H_{SB}}. \tag{4}$$

Note that by the P operator we mean $P \otimes I_B$. Using the relation $P^\dagger \exp(-i\tau H_{SB})P = \exp(-i\tau P^\dagger H_{SB} P)$ it is easy to see that if

$$P^\dagger H_{SB} P = -H_{SB} \tag{5}$$

Then the total unitary over the time period 2τ is simplified as

$$U_{SB}(2\tau^+) = e^{-i\tau P^\dagger H_{SB} P} e^{-i\tau H_{SB}} = e^{i\tau H_{SB}} e^{-i\tau H_{SB}} = I. \tag{6}$$

This means that if we look at the dynamics of the system and bath over a period of time 2τ the total unitary is the trivial Identity meaning no effective coupling between system and bath. This is somehow surprising that we have cancelled the non-unitary effect of the system-bath couplings by just applying unitary transformations on the system only. However choosing the right pulse P requires knowledge of the system-bath interaction H_{SB}.

In this homework we will learn how a universal set of pulses can be applied such that without any knowledge of H_{SB}, the effective system-bath coupling becomes weaker (no complete cancellation).

Question: A single qubit is coupled to bath with an interaction of general form $H_{SB} = X \otimes B_x + Y \otimes B_y + Z \otimes B_z$ with the Pauli operators $\{X, Y, Z\}$ and some bath operators $\{B_x, B_y, B_z\}$. Here we are looking at the average dynamics on four time intervals τ. At the end of each interval the following pulses are applied in the given order $\{X, Z, X, Z\}$ as depicted if Fig.(1). Show that at the end of the fourth interval $(T = 4\tau)$ the total system-bath unitary is

$$U_{SB}(T = 4\tau^+) = e^{-iTI_S \otimes B_0 + T^2(X \otimes B'_x + Y \otimes B'_y + Z \otimes B'_z) + O(T^3)} \tag{7}$$

where $O(T^3)$ means higher orders of T. The term $I_S \otimes B_0$ means that the system-bath coupling is cancelled to the first order of $T = 4\tau$. Therefore if we apply pulses X, Z, X, Z at very short intervals τ we will get a relatively good decoupling of the system and bath.

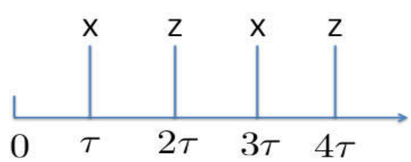

Figure 1: A 4-pulse dynamical decoupling cycle