EE290O
Homework 1: Linear Algebra

Problem 1: Find the eigenvalues and eigenvectors of the matrix $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and argue that this matrix does not have a diagonal representation.

Problem 2: (Principal Submatrix) For a matrix $M_{n\times n}$, a principal submatrix $M(S)$ is formed by choosing a subset S of $\{1, 2, \ldots, n\}$ and defining $M(S)$ as the matrix formed by deleting all the rows of M whose indices are not in S, and all the columns of $M(S)$ whose indices are not in S.
For example $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is principal submatrix of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 3 & 1 & 0 \end{pmatrix}$.

Prove that a principal submatrix of a positive matrix is also positive.

Problem 3: (Pauli Matrices) Three matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in addition to the identity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are the most commonly used basis to describe the space of operators on a two-dimensional Hilbert space. Consider a 3-dim vector with length a and direction $\hat{n}$: $\vec{a} = a\hat{n} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$. Consider the vector of Pauli matrices $\vec{\sigma} = \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}$, and the inner product $\vec{a}.\vec{\sigma} = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$.

Show that $e^{i\vec{a}.\vec{\sigma}} = I\cos(a) + i(\vec{n}.\vec{\sigma})\sin(a)$. You may want to use the identity $\sigma_a\sigma_b = \delta_{ab}I + i\sum_c \epsilon_{abc}\sigma_c$ where ϵ_{abc} is Levi-Civita symbol.

Problem 4: (Completely Positive map) Defined on a Hilbert space $\mathcal{H}$, A linear transformation Φ mapping an operator/matrix ρ to another operator/matrix $\Phi(\rho)$ is called completely positive (CP) iff it can be represented as $\Phi(\rho) = \sum_\alpha E_\alpha \rho E_\alpha^\dagger$ for a set of operators E_α. Suppose $\{|i\rangle\}$ is a complete orthonormal basis of the Hilbert space $\mathcal{H}$. Show that if Φ is a CP map then the matrix

$$\sum_{ij} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|) \tag{1}$$

defined on the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}$ is a positive matrix ($|i\rangle$ and $|j\rangle$ are the basis vectors.). Hint: Use the vector $|\psi\rangle = \sum_i |i\rangle$.