In compressible fluid mix of components A and B.

Densities are \( \rho_A \) and \( \rho_B \).

\[
\frac{A}{V} = T \left[ \frac{\rho_A}{T} \ln \left( \frac{p}{\rho_A} \right) + \frac{\rho_B}{T} \ln \left( \frac{p}{\rho_B} \right) \right] + \frac{\rho_A}{\rho_B} \rho_B T \rho_A + \alpha \rho_A \rho_B \rho_B^2
\]

\( \alpha, \gamma, \gamma \) are constants, \( \rho_A + \rho_B = \rho \)

(a) Find the pressure as a function of \( T \) and \( \rho_A \), and the constants \( \alpha, \gamma, \gamma \), and \( \rho \).

\[
-\rho = \left( \frac{\partial A}{\partial V} \right)_{T, \rho_A, \rho_B} \cdot \frac{1}{T} \left( \frac{\partial A}{\partial V} \right)_{T, \rho_A, \rho_B} = -\frac{\rho}{T}
\]

Now let \( q = \frac{A}{TV} \)

\[
\frac{\rho}{T} = -\left( \frac{\partial A}{\partial V} \right)_{T, \rho_A, \rho_B} \frac{1}{TV} = -q - V \left( \frac{\partial q}{\partial V} \right)_{T, \rho_A, \rho_B} = -q - V \left( \frac{\partial q}{\partial \rho_A} \right)_{T, \rho_A, \rho_B} \left( \frac{\partial \rho_A}{\partial \rho_A} \right)_{T, \rho_A, \rho_B}
\]

\[
\frac{\rho}{T} = -q + \rho_A \left[ \gamma + \rho_B \ln \left( \frac{\rho_A}{\rho_B - \rho_A} \right) + \frac{\alpha}{T} \left( \rho - 2 \rho_A \right) \right]
\]

Plug in and do math

\[
= -\left( \rho_A \ln(\rho_A) + (\rho - \rho_A) \ln((\rho - \rho_A) \gamma) + \frac{\alpha}{T} \rho_A (\rho - \rho_A) + Y \right) + \rho_A \gamma + \rho_A \ln \rho_A - \rho_A \ln(\rho_A) + \frac{\alpha}{T} \left( \rho_A \rho - 2 \rho_A^2 \right)
\]

\[
\rho = -T \rho_A \ln \left( \frac{\rho_A - \rho}{\rho} \right) - \frac{\alpha}{T} \rho_A^2
\]
(b) Find the chemical potential of species A as a function of $T$ and $P_A$ and the constants $x$, $y$, $v$, and $T$.

$$\frac{\mu_A}{T} = \frac{1}{T} \left( \frac{\partial A}{\partial N_A} \right)_{T, V, N_B} = \left( \frac{\partial (A + TV)}{\partial (N_A)} \right)_{T, V, N_B} = \left( \frac{\partial A}{\partial P_A} \right)_{T, V, N_B}$$

$$\left( \frac{\partial A}{\partial P_A} \right)_{T, V, N_B} = y + \ln \left( \frac{P_A}{P - P_A} \right) + \frac{\alpha}{T} \left( P - 2P_A \right)$$

$$\mu_A = TVy + TV \ln \left( \frac{P_A}{P - P_A} \right) + \alpha \left( P - 2P_A \right)$$

(c) Show that for a range of temperatures and densities the expression violates stability.

If \( \left( \frac{\partial \left( \frac{\mu_A}{T} \right)}{\partial P_A} \right)_{T, V} < 0 \), that is an instability.

$$\left( \frac{\partial \left( \frac{\mu_A}{T} \right)}{\partial P_A} \right)_{T, V} = \frac{1}{T} \left( \frac{\partial A}{\partial N_A} \right)_{T, V, N_B} = \frac{1}{V} \left( \frac{\partial}{\partial N_A} \left( \frac{\partial A}{\partial N_A} \right) \right)_{T, V, N_B} = \frac{1}{V} \frac{\partial^2 A}{\partial N_A^2} \left( \frac{\partial A}{\partial N_A} \right)_{T, V, N_B} = \frac{V}{T} \frac{\partial^2 A}{\partial N_A^2} \left( \frac{\partial A}{\partial N_A} \right)_{T, V, N_B} \Rightarrow 0 \geq 0$$

At the boundary:

$$\frac{1}{P_A} + \frac{1}{P - P_A} = \frac{2\alpha}{T}$$

$$\frac{1}{P_A} \left( \frac{P - P_A}{P} \right) = \frac{2\alpha}{T}$$

$$P_T = 2 \alpha \frac{P_A}{P}$$

$$T_{boundary} = \frac{2 \alpha P_A (P - P_A)}{P}$$

$$T_b = 2 \alpha \left( \frac{P_A - P_A^2}{P} \right)$$
(d) Phase separation will occur to "bridge" the instability. What is the critical temperature and density of species A? Look at a $T-f_{\Lambda}$ diagram.

The critical point occurs when $T_{\text{boundary}}$ is at its max.

\[
\frac{\partial T_{\text{b}}}{\partial (f_{\Lambda}/p)} = 0
\]

\[T_{\text{b}} = 2\alpha fX(1-X) \quad \text{where} \quad X = f_{\Lambda}/p\]

\[
\left(\frac{\partial T_{\text{b}}}{\partial X}\right)_p = \left[\frac{\partial}{\partial X} \left(2\alpha fX - 2\alpha fX^2\right)\right]_p
\]

\[= 2\alpha f - 4\alpha fX = 0\]

\[\frac{f_{\Lambda}}{p} = \frac{1}{2}\]

The critical value of $f_{\Lambda} = \frac{1}{2} f$

\[T_{\text{c}} = \frac{2\alpha f}{2}\]
When the phases are in coexistence we have equality of the intensive variables:

\[ M(x^0, T) = M(x^0, T) \]

\[ M(x, T) = T \gamma + T \ln \left( \frac{x}{1-x} \right) + \alpha p (1-2x) \]

This means that

\[ T \ln \left( \frac{x^{(2)}}{1-x^{(2)}} \right) - \alpha p x^{(2)} = T \ln \left( \frac{x^{(0)}}{1-x^{(0)}} \right) - \alpha p x^{(0)} \]

We also have equal pressures in each phase.

\[ p^{(0)} = p^{(2)} \]

\[ p(x, T) = -T \beta \ln \left[ \frac{(1-x)p}{\alpha} \right] - \alpha p x^2 \]

Plugging in we arrive at

\[ T \ln \left( \frac{1-x^{(0)}}{1-x^{(2)}} \right) - \alpha p x^{(0)} = T \ln \left( \frac{1-x^{(2)}}{1-x^{(0)}} \right) - \alpha p x^{(2)} \]

(F) For conditions close to the critical temperature and density.

\[ |\Delta p| \sim (T_c - T)^{0.5} \]

Consider the boundary \( \frac{\Delta p}{p} = \frac{p_{21} - p_{22}}{p} \). This function has two values for \( p_{21}/p \)

\[ T = \frac{2 \alpha}{p} (p_{21}p_{22} - \alpha p^2) \]

Solve as a quadratic

\[ p_{21} = \frac{p \pm \sqrt{\left(p^2 - 2Tf/p \right)^2}}{2} \]

\[ 1|\Delta p| = \sqrt{p^2 - 2Tf/p} \]

Again, \( T_c = \alpha/\lambda \Rightarrow p = \frac{2T_c}{\alpha} \)

\[ 1|\Delta p| = \sqrt{\frac{2T_c}{\alpha} p - \frac{2T_c}{\alpha} p} = \sqrt{2T_c (T_c - T)} \]

\[ 1|\Delta p| \sim (T_c - T)^{1/2} \]
Two-component gas of non-interacting classical structureless particles with mass \( m_A \) and \( m_B \) at a temperature \( T \).

(a) Calculate exactly the grand canonical partition function.

In the canonical ensemble we would have

\[
Q(N_A, N_B, V, T) = \frac{Q_A}{N_A!} \frac{Q_B}{N_B!}
\]

For non-interacting, structureless particles, we have only energy from translational motion.

From lecture we have

\[
\mathcal{Q} = \left( \frac{2\pi m}{\hbar^2} \right)^{3/2} V
\]

\[
\Xi = \sum_{N_A=0}^{\infty} \sum_{N_B=0}^{\infty} \frac{Q_A}{N_A!} \frac{Q_B}{N_B!}
\]

\[
= \left[ \sum_{N_A=0}^{\infty} \frac{(2\pi m_A V e)^{N_A}}{N_A!} \right] \left[ \sum_{N_B=0}^{\infty} \frac{(2\pi m_B V e)^{N_B}}{N_B!} \right]
\]

Note that

\[
e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots
\]

\[
\Xi = e^{2\pi m_A V e} e^{2\pi m_B V e} = e^{(2\pi m_A V e)^{3/2}} V e^{(2\pi m_B V e)^{3/2}} V e
\]

\[
V \left( \frac{2\pi}{\hbar^2} \right)^{3/2} \left[ \frac{3}{2} \beta V e^{(2\pi m_A V e)^{3/2}} + \frac{3}{2} \beta V e^{(2\pi m_B V e)^{3/2}} \right]
\]

\[
= e^{\Xi} = e^{Vz}
\]

with \( z = \left( \frac{2\pi}{\hbar^2} \right)^{3/2} \left[ \frac{3}{2} \beta V e^{(2\pi m_A V e)^{3/2}} + \frac{3}{2} \beta V e^{(2\pi m_B V e)^{3/2}} \right] \)

(b) Find the pressure, \( P \).

\[
P = \frac{1}{V} \frac{\partial \mathcal{Q}}{\partial V}
\]

\[
P = \frac{1}{V} \frac{\partial \mathcal{Q}}{\partial V} = \frac{1}{V} \frac{\partial \Xi}{\partial V}
\]

\[
P = \frac{1}{V} \Xi = \frac{1}{V} \left[ \frac{3}{2} \beta V e^{(2\pi m_A V e)^{3/2}} + \frac{3}{2} \beta V e^{(2\pi m_B V e)^{3/2}} \right]
\]

\[
P = \frac{1}{V} \Xi = \frac{z}{\beta}
\]
Note that we need the answer in terms of \( \Gamma_A \) and \( \Gamma_B \).

\[
\langle N_A \rangle = \left( \frac{\partial H}{\partial \beta N_A} \right)_{B,V} = \sqrt{\frac{2}{\beta h^2}} \left[ \frac{2\pi}{\beta h^2} \right]^{3/2} \frac{3/2}{M_A} e^{\beta M_A} = \sqrt{\frac{2\pi}{\beta h^2}} \frac{3/2}{M_A} e^{\beta M_A}
\]

\[
\Gamma_A = \frac{\langle N_A \rangle}{\sqrt{\frac{2}{\beta h^2}}} = \left( \frac{2\pi M_A}{\hbar^2} \right)^{3/2} e^{\beta M_A}
\]

\[
\Gamma_B = \left( \frac{2\pi M_B}{\hbar^2} \right)^{3/2} e^{\beta M_B}
\]

Note that \( \Gamma = \Gamma_A + \Gamma_B \).

\[
\Gamma = \frac{\Gamma_A + \Gamma_B}{\beta}
\]

(c) For a 1 CC of a 50/50 mixture of A and B at STP, compute the relative root mean square of the density fluctuations.

\[
\left[ \frac{\langle (\delta p)^2 \rangle}{\langle p^2 \rangle} \right]^{1/2}
\]

\[
\langle (\delta p)^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2 = \langle (p_A + p_B)^2 \rangle - \langle p_A + p_B \rangle^2
\]

\[
= \langle p_A^2 + 2p_A p_B + p_B^2 \rangle - \left[ \langle p_A^2 + 2p_A p_B + p_B^2 \rangle \right]^2
\]

\[
= \langle (\delta p_A)^2 \rangle + \langle (\delta p_B)^2 \rangle
\]

Hence

\[
\frac{\langle (\delta p)^2 \rangle}{\langle p^2 \rangle} = \frac{\langle (\delta p_A)^2 \rangle + \langle (\delta p_B)^2 \rangle}{\langle p_A + p_B \rangle^2}
\]

\[
\langle (\delta p_A)^2 \rangle = \frac{\partial}{\partial \beta N_A} \langle N_A \rangle = V \Gamma_A = \langle N_A \rangle \quad \text{and} \quad \langle (\delta p_B)^2 \rangle = \langle N_B \rangle
\]

\[
\frac{\langle (\delta p)^2 \rangle}{\langle p^2 \rangle} = \frac{\langle N_A \rangle + \langle N_B \rangle}{\langle N_A \rangle + \langle N_B \rangle} = \frac{1}{\langle N_A \rangle + \langle N_B \rangle}^{1/2} = \frac{1}{\langle N \rangle^{1/2}}
\]
\[ \langle N \rangle = \frac{pV}{k_B T} \]

At 573 \, \text{K}, 1 \, \text{cc},

\[
\langle N \rangle = \left( \frac{101325 \, \text{N}}{\text{m}^2} \right) \left( 1 \times 10^{-6} \, \text{m}^3 \right) = 7.7 \times 10^{19} \, \text{atoms}
\]

\[
\sqrt{\langle \delta N^2 \rangle} = 1.9 \times 10^{-10}
\]

\[ \frac{(\delta N)^2}{\langle N \rangle} \]

(d) What is the probability of observing a density fluctuation in 1 cc of a 50\% mix of gas at 573 \, \text{K} for which the instantaneous density differs from the mean by 1 in 10^6?

\[
\ln \frac{P(N,n)}{P(N)} = \frac{1}{2} (\delta N)^2 \left( \frac{\partial^2 \ln P(N)}{\partial N^2} \right)_{N=\langle N \rangle}
\]

We want the conditional probability

\[
\frac{P(N)}{P(N)} = e
\]

\[
(\delta N)^2 = \left( 10^{-6} \langle N \rangle \right)^2
\]

\[
\frac{\partial^2 \ln P(N)}{\partial N^2} \bigg|_{N=\langle N \rangle} = \left[ \text{For a Gaussian} \right] = -\frac{1}{\langle (\delta N)^2 \rangle} = -\frac{1}{\langle N \rangle}
\]

\[
\frac{P(N)}{P(N)} = e^{-\frac{1}{2} (\delta N)^2 / \langle N \rangle} = e^{-10^{-12} \langle N \rangle^2 / \langle N \rangle} = e^{-10^{-12} \langle N \rangle^2 / \langle N \rangle} = 1.3 \times 10^{-7}
\]
we have \( N \) Ising spins arranged along a ring. The energy is given by

\[
H = -J \sum_{j=1}^{N} \sigma_j \sigma_{j+1}
\]

Let's use the canonical ensemble.

\[
\mathcal{D} = \sum_y e^{-\beta E_y} = \sum_{\sigma = \sigma_1 \cdots \sigma_N} e^{-\beta \sum_j J \sigma_j \sigma_{j+1}} = \sum_{\sigma = o} e^{-\beta \sum_j J \sigma_j \sigma_{j+1} \sigma_{j+1}}
\]

\[
= \sum_{\sigma = o} \left( e^{-\beta J \sigma \cdot \sigma_{j+1}} \right)^N = \left( e^{-\beta J} + e^{\beta J} \right)^N = \left[ 2 \cosh (\beta J) \right]^N
\]

Now,

\[
-\beta \mathcal{A} = \mathcal{A}_n \mathcal{D} = N \mathcal{A}_n \left[ 2 \cosh (\beta J) \right]
\]

\[
\mathcal{A} = -\frac{N}{\beta} \mathcal{A}_n \left[ 2 \cosh (\beta J) \right]
\]

\[
\langle E \rangle = -\frac{\partial \mathcal{A}}{\partial \beta} = \frac{-N}{2} \frac{\sinh (\beta J)}{\cosh (\beta J)} J = -NJ \tanh (\beta J)
\]

\[
\mathcal{C}_x = \frac{\partial \langle E \rangle}{\partial T} = \frac{\partial}{\partial T} \left[ -NJ \tanh \left( \frac{J}{k_B T} \right) \right] = \frac{NJ^2}{k_B T^2} - \frac{1}{\cosh^2 \left( \frac{J}{k_B T} \right)}
\]
Magnitude of the spontaneous magnetization of a 1-D Ising model at zero temperature.

The partition function for the 1-D case w/o magnetic field is given by
\[ \Omega(\beta, N, 0) = \left[ 2 \cosh (\beta J) \right]^N \]

At \( H=0 \)

Note from the discussion on p122 in Chandler that to create a disordered state from an ordered state requires very little energy. In fact, for this system, \( T = 2J \frac{J}{Nk_B} \) which is near zero for large \( N \).

At \( T=0 \) we expect all spins to point in one direction, either all up or all down.

\[ \langle E \rangle = \mu H \langle m \rangle \] so that \( E = \pm \mu H N \) for \( N \) spins.

\[ \langle E \rangle = -J \frac{\partial \ln \Omega}{\partial \beta} = -N \beta J \tanh(\beta J) = \pm \mu H N \]

As \( T \to 0 \), \( \beta \to \infty \) and \( \tanh(\infty) = 1 \)

\[ \langle E \rangle = \pm \mu H N = NJ \]

Hence \( J = \pm \mu H \)

So as \( T \to 0 \) the magnetization changes from \( \langle m \rangle = 0 \) to \( \langle m \rangle = \pm N \)

So spontaneous magnetization in the 1-D Ising model only happens at \( T \to 0 \)
Problem 5.4 on page 124 in Chandler. The point of this problem is to show that the Ising magnet model maps onto the model of the lattice gas, and this is using for thinking about problems not just involving magnets.

Note that for the Ising model

\[ E_I = -\sum_{i=1}^{N} h_i s_i - J \sum_{i<j} s_i s_j \quad \text{and} \quad \Omega = \sum_{\{s_i\} = \uparrow} \exp \left[\beta N h_i \sum_{i=1}^{N} s_i + \beta J \sum_{i<j} s_i s_j \right] \]

where \( \sum_{i<j} \) refers to a summation over nearest neighbor pairs.

To map this onto the lattice gas we let

\[ s_i = 2n_i - 1 \]

Then we have

\[ E_I = -m H \sum_{i=1}^{N} (2n_i - 1) - J \sum_{i<j} (2n_i - 1)(2n_j - 1) \]

\[ = -2m H \sum_{i=1}^{N} n_i + m N H - 4J \sum_{i<j} n_i n_j + 2J \sum_{i<j} (n_i + n_j) - J \sum_{i<j} n_i n_j \]

This is the same as \( Z \sum_{i=1}^{N} n_i \) as \( Z \sum_{i=1}^{N} n_i \) as \( N \frac{Z}{2} \)

Now, for a lattice gas we have

\[ E = \sum_{\{n_i\} = 0,1} \exp \left\{ \beta m H \sum_{i=1}^{N} n_i + \beta E \sum_{i<j} n_i n_j \right\} \]

Thus

\[ \Omega_{\text{Ising}} = \exp \left[ \beta N (m H - J N/2) \right] \]

\[ m = 2m H - 2J z \]

\[ E = 4J \]

\[ Z \] is the number of nearest neighbors to any given lattice site.
The point of these calculations involving mean field theory is to make approximations to the 2-d Ising model that allow us to obtain results easily. Neighboring particles form an average "field" which acts on the tagged particle or spin. The main result is the plot in Fig 5.9 and the critical temperature given by

$$T_c = \frac{2DJ}{kB}$$

where $D$ is dimensionality (e.g. 2).

(a) We start with

$$\phi = \sum_{N_t} \sum_{N_{tt}} \exp \left(-\beta E(N_t, N_{tt})\right)$$

Note: $E(N_t, N_{tt}) = (-4J)N_{tt} + 2(2J - H\mu)N_t - \left(\frac{2J}{2} + H\mu\right)N$

$$\phi = e^{-\beta (\frac{2J + H\mu}{2})N} \sum_{N_t=0}^{\infty} e^{-2\beta (2J - H\mu)N_t} \sum_{N_{tt}=0}^{\infty} g(N_t, N_{tt})$$

where $g(N_t, N_{tt}) = \# of configurations for a given values of $N_t$ and $N_{tt}$

$$\frac{N_{tt}}{N} = fraction that are "+" \quad = \frac{1}{2} (\ell_{+1}) \quad (-1 \leq \ell \leq 1)$$

$$\frac{2N_{tt}}{N} = fraction of bonds that are "+++" \quad = \frac{1}{2} (\ell_{+1} \ell_{+1}) \quad (-1 \leq \ell \leq 1)$$

$$N_{tt} = \frac{N}{4} (-\ell_{+1})$$

we now need to re-write the energy in terms of $\ell$ and $\ell_{+1}$.

$$E = \frac{-4J}{4} \left(\ell_{+1}\right) + 2(2J - H\mu)(\frac{1}{2})(\ell_{+1}) - \left(\frac{2J}{2} + H\mu\right)$$

$$= -\frac{J}{2} \left[2\ell + 2 - 2L - 2\ell_{+1}\right] - H\mu \left[\ell_{+1} - 1\right]$$

$$= -\frac{J}{2} \left[2\ell - 2L + 1\right] - H\mu \left[\ell\right]$$
Next employ the Bragg-Williams Approximation.

\[ \frac{W_{++}}{\frac{1}{2}ZW} = \frac{(N_{+})^2}{N} \]

\[ W_{++} = \frac{1}{2} (L+1) = \left[ \frac{1}{2} (L+1) \right]^2 \]

\[ \sigma = \frac{1}{2} (L+1)^2 - 1 \]

\[ = \frac{L^2}{2} + L - \frac{1}{2} \]

Plugging into our expression for \( E/W \)

\[ E \frac{1}{N} = - \frac{5ZL^2}{2} - \frac{N+1}{2}HL \]

Now \( E \) is a function only of \( L \) and \( N \).

\[ \Omega = \sum_{\text{all } L} \sum (L) \exp \left( -\beta E(L) \right) \]

\[ \sum (L) = \sum (N_{+}) = \frac{N!}{N_{+}! (N-N_{+})!} \]

Hence

\[ \Omega = \sum_{L=-1}^{+1} \frac{N!}{N_{+}! (N-N_{+})!} e^{\left( \frac{5ZL^2}{2} + \frac{N+1}{2}HL \right)N} \]

\[ N_{+} = \frac{N}{2} (1 + \lambda) \quad N - N_{+} = \frac{N}{2} (1 - \lambda) \]

\[ \Omega = \sum_{L=-1}^{+1} \frac{N!}{\left[ \frac{N}{2} (1 + L)! \right] \left[ \frac{N}{2} (1 - L)! \right]} e^{\left( \frac{5ZL^2}{2} + \frac{N+1}{2}HL \right)N} \]
(b) What is $\tilde{\Sigma}$?

\[ L = \frac{2N_+}{N} - 1 \]

so $L$ is like a the fraction of sites with a "+$" spin. It is therefore related to the net magnetization.

We assume that the partition sum is dominated by one term in the summation -- the term that is weighted most heavily (has the most favorable energy).

Thus

\[ \Theta \bigg|_{L=\tilde{\Sigma}} = \frac{N!}{\left(\frac{N}{2}\right)! \left(\frac{N}{2}\right)!} e^{N \frac{1}{2} \left(\tilde{\Sigma} \frac{L^2}{K} + B_n H L\right) N} \]

\[ = \frac{N!}{\left(\frac{N}{2}\right)! \left(\frac{N}{2}\right)!} e^{N \frac{1}{2} \left(\tilde{\Sigma} \frac{L^2}{K} + B_n H L\right) N} \]

\[ = \Theta e^{N \frac{1}{2} \left(\tilde{\Sigma} \frac{L^2}{K} + B_n H L\right) N} \]

To find the max or min in $N$, we should find the max or min in $\ln \Theta$.

\[ \ln \Theta = N \ln N - N - \left[ N \left(1 + \frac{1}{2}\right) \ln \left[\frac{N}{2} \left(1 + \frac{1}{2}\right)\right] - \frac{N}{2} \left(1 + \frac{1}{2}\right) \ln \left[\frac{N}{2} \left(1 - \frac{1}{2}\right)\right] \right] \]

\[ - \frac{N}{2} \left(1 - \frac{1}{2}\right) \]

\[ + \left(\frac{N}{2} \frac{L^2}{K} + B_n H L\right) N \]

Do some canceling and divide by $N$.

\[ \frac{1}{N} \ln \Theta = \left[ \frac{1}{2} \left(1 + \frac{1}{2}\right) + \frac{1}{2} \left(1 - \frac{1}{2}\right) \right] \ln N - \frac{N}{2} \left(1 + \frac{1}{2}\right) \ln \left[\frac{N}{2} \left(1 + \frac{1}{2}\right)\right] \]

\[ - \frac{N}{2} \left(1 - \frac{1}{2}\right) \ln \left[\frac{N}{2} \left(1 - \frac{1}{2}\right)\right] + \frac{N}{2} \frac{L^2}{K} + B_n H L \]

we find $\tilde{\Sigma}$ by calculating

\[ \frac{d}{d\tilde{\Sigma}} (\text{RHS}) = 0 \]

\[ \beta \left(\frac{1}{2} Jzz \tilde{Z} + B_n H L\right) - \frac{1}{2} \left(1 + \frac{1}{2}\right) \ln \left[\frac{1}{2} \left(1 + \frac{1}{2}\right)\right] + \frac{1}{2} \left(1 + \frac{1}{2}\right) \ln \left[\frac{1}{2} \left(1 - \frac{1}{2}\right)\right] \]

\[ + \left(1 - \frac{1}{2}\right) \frac{1}{\frac{1}{2} \left(1 - \frac{1}{2}\right)} \left(1 - \frac{1}{2}\right) = 0 \]
This gives
\[ \beta (J x L + H x L) - \frac{1}{2} \left( \frac{c_1 (1 + \bar{L})}{1 - \bar{L}} \right) = 0 \]

\[ \ln \left( \frac{1 + \bar{L}}{1 - \bar{L}} \right) = 2 \beta \mu H + 2 \beta J x L \]

Note that \[ \frac{\ln c_1}{2 \beta \mu H + 2 \beta J x L} \]
\[ \frac{1 + \bar{L}}{1 - \bar{L}} = e \]
\[ \frac{1 + \bar{L}}{1 - \bar{L}} = e^{\frac{2c_1}{1 - \bar{L}}} \]
\[ e^{\frac{2c_1}{1 - \bar{L}}} = \frac{c_1}{\bar{L}} \]
\[ \bar{L} = e^{\frac{2c_1}{1 - c_1}} = \frac{c_1}{c_1 + e^{-c_1}} = \tanh (c_1) \]
\[ \bar{L} = \tanh (\beta H x L + \beta J x L) \] (3)

(c) we can't solve equation (3) for \( L \) explicitly, but we can look at it graphically. See Fig 5.7.

For the case of \( H = 0 \)
\[ \tanh (\beta J x L) \]
\[ \beta J x L > 1 \]
\[ \text{This is the line} \]
\[ \bar{L} = \bar{L} \]
\[ \beta J x L < 1 \]

For \( \beta J x L > 1 \) we see that the equation has a solution for non-zero \( \bar{L} \). You can also find this numerically rather than graphically.

\[ \beta J x L = \frac{1}{k_B T}, \quad T_c = \frac{J x L}{k_B} \quad \text{for} \quad \beta J x L = 1 \]

Having a non-zero critical temperature means that for \( T_c \) we expect to see the Ising magnet magnetized! This is the phenomena of broken symmetry.