Numerically exact long-time magnetization dynamics at the nonequilibrium Kondo crossover of the Anderson impurity model

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We investigate the dynamical and steady-state spin response of the nonequilibrium Anderson model to magnetic fields, bias voltage, and temperature using a numerically exact method combining a bold-line quantum Monte Carlo technique with a memory function formalism. We obtain converged results in a range of previously inaccessible regimes, in particular calculating the spin dynamics for a range of temperatures down to the crossover to the Kondo domain. We provide predictions for nonequilibrium phenomena, including nonmonotonic temperature dependence of observables at high bias voltage and oscillatory quench dynamics at high magnetic fields.

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Strongly correlated open quantum systems appear in a wide variety of physical situations, including quantum dots in semiconductor heterostructures,12 molecular electronics,3,4 and the dynamics of cold atoms.5 These systems consist of a finite, interacting region coupled to a continuous set of noninteracting “bath” or “lead” states which may be maintained at differing thermodynamic states. It is natural to describe open systems in terms of quantum impurity models, which have been used in the description of magnetic impurities in metals,6 the adsorption of atoms on a surface7 and as auxiliary problems in the dynamical mean field approximation to extended lattice systems.8 More recently, quantum impurity models have also been of interest in the nonequilibrium context of mesoscopic transport9,10 and of nanosystems coupled to broad leads.2

While attempts are being made to connect nonequilibrium physics to equilibrium concepts,11 the nonequilibrium steady-state properties of correlated quantum systems continue to present a formidable challenge to our theoretical understanding. The main difficulty is that a rigorous evaluation of the long-time and steady-state properties requires an accurate time propagation, starting from some known initial state and reaching all the way to the steady state. When this relaxation occurs quickly, a range of powerful semianalytical12–14 and numerical methods15–25 are applicable. However, dynamics in strongly correlated systems may exhibit a separation of time scales—for example, the spin-relaxation dynamics in the Kondo regime of a quantum dot are orders of magnitude slower than those of the corresponding charge relaxation. Existing theoretical methods are unable to resolve these time scales reliably in the general case (though progress in analytical methods26–28 can teach us much about generic aspects of the problem).

In this paper we show that a combination of bold-line diagrammatic Monte Carlo methods24,29 and the memory-function approach25 enables us to significantly extend the time regime accessible to numerical simulation and can, in some cases, access steady-state information within the Kondo regime (though not deep within the strong-coupling regime).

The method is numerically exact and provides unbiased error estimates. While the calculations presented here are for the single impurity Anderson model, a minimal model for strong interactions in the presence of baths, the methodology is applicable to any quantum impurity model.30

The Anderson impurity model is defined by the Hamiltonian

$$H = H_S + H_B + V,$$

where $H_S$ describes the interacting system (or dot) part, $H_B$ the noninteracting bath (or leads) part, and $V$ the system-bath coupling part:

$$H_S = \sum_{i=\uparrow \downarrow} \epsilon_i d_i^\dagger d_i + U d_i^\dagger d_i d_j^\dagger d_j,$$

$$H_B = \sum_{k,i=\uparrow \downarrow} \epsilon_{ik} a_{ik}^\dagger a_{ik},$$

$$V = \sum_{k,i=\uparrow \downarrow} t_{ik} a_{ik}^\dagger + t_{ik}^* a_{ik} d_j^\dagger.$$
same notation defining $L_V$ and $L_H$: $\vartheta(t)$ is an initial correlation term which vanishes for factorized initial conditions $\rho(0) \equiv \rho_A \otimes \sigma(0)$; and $\rho_B$ is the initial bath density matrix. The memory kernel $\kappa$ may be obtained by solving

$$\kappa(t) = i\hbar \Phi(t) - \Phi(t) L_S + \frac{i}{\hbar} \int_0^t dt' \Phi(t - t') \kappa(t),$$

where the superoperator $\Phi(t) \equiv \text{Tr}_B [L_V e^{-\frac{i}{\hbar} L_O t} \rho_B]$ must in general be obtained from a many body computation whose expense rapidly increases as $t$ increases. Evaluation of $\Phi(t)$ for $t$ up to a cutoff time $t_c$ allows an exact evaluation of $\kappa$ and thus $\sigma$ for $t < t_c$. Setting $\kappa(t > t_c) = 0$ defines the cutoff approximation, whose convergence may be monitored from the dependence of results on $t_c$ as $t_c$ is increased.

In the case of the Anderson impurity model, Ref. 25 has shown that if one is only interested in evaluating the diagonal elements of the density matrix, all the supermatrix elements $\Phi_{ij,qq} \equiv \langle i|\langle j|\Phi|q\rangle|q\rangle$ of $\Phi$ having $i \neq j$ or $q \neq q'$ can be set to zero, with the remaining elements determined by

$$\Phi_{ii,qq} = \delta_{0}(\rho^{(1)}_{qq} + \rho^{(3)}_{qq}) + \delta_{1}(\rho^{(2)}_{qq} - \rho^{(3)}_{qq}) + \delta_{2}(\rho^{(1)}_{qq} + \rho^{(4)}_{qq}) + \delta_{3}(\rho^{(2)}_{qq} - \rho^{(4)}_{qq}),$$

where $A^{(1)}_k = t_{\uparrow} d_{\uparrow}^{\dagger} d_{\uparrow}^{\dagger} a_k^{\dagger}$, $A^{(2)}_k = t_{\downarrow} d_{\downarrow}^{\dagger} d_{\downarrow}^{\dagger} a_k^{\dagger}$, $A^{(3)}_k = t_{\uparrow} d_{\uparrow}^{\dagger} d_{\downarrow}^{\dagger} a_k^{\dagger}$, and $A^{(4)}_k = t_{\downarrow} d_{\uparrow}^{\dagger} d_{\downarrow}^{\dagger} a_k^{\dagger}$.

The evaluation of the $\rho^{(m)}_{qq}(t)$ has previously been performed with real time path integral Monte Carlo (RT-PIMC) methods,\textsuperscript{15,25,35} revealing that, in the presence of strong electronic correlations, the memory kernel may develop long tails. Near the Kondo regime this effect becomes particularly pronounced, making it impossible to converge the cutoff approximation and highlighting the need for methods able to obtain $\kappa$ for longer times. Here we show that the problem can to a large extent be solved by using the bold expansion for impurity models,\textsuperscript{28} a technique related to bold-line methods for lattice systems.\textsuperscript{36-38} The bold expansion is based on a stochastic Monte Carlo sampling of diagrammatic corrections to the propagators obtained from an infinite partial summation, rather than a sampling of all diagrams. Intrinsically to the method is a tradeoff between complexity and memory requirements of the initial partial resummation and the number of additional diagrams which must be stochastically evaluated. The resulting procedure converges at lower expansion order than RT-PIMC and greatly reduces the severity of the dynamic sign problem, in practice more than doubling the accessible time scales relative to a bare expansion and allowing reliable estimates of the memory kernel down to much lower temperatures. This allows access to dynamics exhibiting arbitrarily long time scales at those temperatures.

In the nonequilibrium case, diagrams must be evaluated on the Keldysh contour. We find it advantageous to employ a “BoldOCA” built on the one-crossing approximation (OCA).\textsuperscript{29,39} Figure 1 illustrates this in diagrammatic terms: the bold-line propagators and vertex functions (which allow for the summation over hybridization lines connecting pairs of times on the two different halves of the Keldysh contour) are obtained from the solution of the OCA equations, and used in an expansion which samples diagrams of all crossing orders. The increase in complexity and memory required to obtain the initial OCA solution (not needed in the bare expansion) is more than compensated for by the reduction in the number of diagrams which must be stochastically sampled. Unbiased error estimates are obtained by jackknife analysis on data from multiple, uncorrelated Monte Carlo runs (typically $5$ to $10$).

For concreteness, we assume a wide, flat band $\Gamma_i \langle E \rangle = \Gamma_i^L \langle E \rangle + \Gamma_i^R \langle E \rangle$ with $\Gamma = \frac{\Gamma_L}{\Gamma_R}$. Here $\bar{e}_c$ and $\bar{v}$ are the band cutoff energy and its inverse cutoff width, and $L$ and $R$ are, respectively, left and right lead indices. We restrict our calculations to the symmetric Anderson impurity model. We apply a magnetic field $h$ in order to generate a nonzero magnetization, setting $\bar{e}_c = -\frac{\bar{V}}{2} \pm \frac{\bar{V}}{2}$ (the formalism is more general and does not rely on this symmetry). We choose $\Gamma$ as our energy unit, and throughout the rest of this paper set $U = 5\Gamma$, $\bar{e}_c = 10\Gamma$, and $\bar{V}/\Gamma = 10$. The initial conditions are determined by assuming an initially decoupled system, having left and right leads thermally equilibrated at a temperature $\beta^{-1}$ and chemical potentials $\mu_L = \frac{\beta}{2}$ and $\mu_R = -\frac{\beta}{2}$, respectively. This defines the lesser and greater hybridization functions $\Delta^L(\omega) = -i [\varphi_{LR}(\omega)] \varphi_{LR}(\omega)$ and $\Delta^\nu(\omega) = i [\varphi_{LR}(\omega)] \varphi_{LR}(\omega)$, which depend on the temperature and chemical potentials through the Fermi occupation function $\varphi_{LR}(\omega) = \frac{1}{1 + e^{\omega - \mu_L}(\omega)}$. At these parameters, the Kondo temperature is given by $\Gamma \beta K \equiv \frac{\beta}{\bar{V}} \simeq 3.4 \times 10^{-3}$.

At zero voltage the method can be benchmarked against the equilibrium magnetization.\textsuperscript{29,41,42} The top left panel of Fig. 2 displays the steady-state magnetization predicted by the proposed method at $V = 0$, plotted against the inverse cutoff time $\frac{1}{\tau_c}$ at several temperatures. For the very small magnetic field $h = 0.01\Gamma$ in Fig. 2, the relative errors are rather large, but considered on the full scale of the magnetization the precision demonstrated here is remarkable. Lower temperatures exacerbate the sign problem, resulting in larger errors and longer computation times.

The effects of taking the system out of equilibrium are illustrated in the lower left panel of Fig. 2. Here a constant temperature $\beta \Gamma = 1$ is maintained while the bias voltage $V$ is varied at $h = 0.11\Gamma$; the numerically exact $V = 0$ result is also shown. This plot clearly illustrates that convergence of the method generally occurs at even shorter times in nonequilibrium conditions—consistent with expectations, equilibrium
exhibits the longest memory while at larger voltages the plateau is reached at shorter cutoff times.

An independent approach to verifying convergence relies on direct examination of individual elements of the memory kernel as a function of time. Several representative elements are displayed at \( h = 0.01\Gamma \) and \( \beta \Gamma = 1 \) in the top right panel of Fig. 2, with the inset highlighting short times. Within the times accessible by BoldOCA, the memory kernel elements go to zero within the numerical accuracy. Below this, on the same time scale and for the same parameters, the time dependence of the three distinct elements of the reduced density matrix \( \sigma \) is plotted for an initially magnetized dot in the lower right panel of Fig. 2. With this initial condition and within the symmetric Anderson impurity model, the diagonal density matrix entries \( \sigma_0 \) and \( \sigma_\tau \), which express charging dynamics, are identical. They both relax so rapidly that their steady-state values could have been obtained to very good accuracy without recourse to memory techniques. The difference in scale between the spin relaxation time of \( \tau \) and the memory decay in the right panel of Fig. 3 shows the time evolution of the magnetization from the generalized magnetic susceptibility

\[
\chi = \int \frac{d\omega}{2\pi} \chi(\omega)
\]

\( V \). The rate for this transfer process is approximately proportional to the lead occupation at the energy difference between the states: \( f(\beta, \Delta E, \mu) = \frac{1}{e^{\beta \Delta E} + 1} \), with \( \Delta E \) equal to half the interaction energy \( \frac{U}{2} \) and \( \mu = \frac{V}{T} \) or \( -\frac{V}{T} \), depending on the lead. \( f \) is therefore an increasing function of temperature for \( V < U \) and a decreasing one for \( V > U \). At small voltages the effect of the population transfer results in a reduction of the magnetization (as expected), while at large voltages the population transfer enhances the intermediate-temperature magnetization. At still larger temperatures, the nonequilibrium effects are washed away and normal thermal suppression of the magnetization occurs.

In Fig. 4 we display the steady-state voltage dependence of the generalized magnetic susceptibility \( \chi \equiv \frac{1}{h} \). At small \( h \) this quantity is \( h \) independent. The top panel shows clearly how the regime in which \( m \) is linear in \( h \) depends on voltage at a constant temperature. The bottom panel of Fig. 4 shows the voltage dependence at different temperatures within the linear regime. One immediately noticeable feature is the decrease of \( \chi \) with increasing \( \beta \) at high voltage, which corresponds to
the nonmonotonic temperature dependence discussed in the bottom panel of Fig. 4. A second interesting feature is the fact that the plots have a simple, Lorentzian-like structure, suggesting that the results may be in a regime accessible to analytical methods based on performing logarithmic corrections around rate equations. In the dotted lines in the bottom panel we show for comparison results obtained by solving the classical rate equations (obtained by perturbation theory to second order in the hybridization). The large discrepancy between the master equation and numerically exact results at $\beta\Gamma = 1$ demonstrates the need to accurately account for non-Markovian effects, as illustrated by the method introduced here.

In conclusion, by unifying numerically exact bold Monte Carlo methods with the exact memory approach we have developed a formally exact formalism free from systematic errors and well suited for the real time solution of nonequilibrium quantum impurity models. In practice, the capabilities of this formalism are unparalleled: the method generates precise, converged results at all time scales, in cases where state-of-the-art approximate methods clearly fail. For the nonequilibrium Anderson impurity model, the formalism performs well even as one enters the Kondo regime, a regime previously inaccessible with accurate numerical methods. It should, however, be noted that the computational difficulty increases at low temperatures; in the strong coupling regime $T_K$ becomes very small, eventually dropping below our accessible temperature range (though at the absolute temperature used in this paper, strong coupling remains accessible).

Our formalism has allowed us to explore the detailed behavior of the nonequilibrium magnetization, and we have made predictions regarding multiscale, oscillatory quenching dynamics at high magnetic fields; the effect of voltage on dynamical relaxation; and population-driven reversal of the magnetization’s temperature dependence at high voltages. These results are obtained at parameters where other currently available methods are not reliable. As the temperature is further lowered, one expects to encounter the formation of Kondo peaks at the chemical potential. How this will affect the behavior described here remains an interesting and open question, and work is currently being carried out to further investigate this issue. Future research will address lower temperatures and a wider variety of observables; it is also worth stressing that both bold techniques and the memory formalism are not specific to the Anderson impurity model, and are expected to have many more applications.

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**References**