\[ m \frac{q_j}{d} = -2 \frac{q_j - q_{j-1}}{d} - 2 \frac{q_{j+1} - q_j}{d} \]

\[ \lim_{d \to 0} \left( \frac{2q_j}{2x} \right) _{x = \frac{d}{2}} - \frac{2q_j}{2x} \right) _{x = -\frac{d}{2}} = \frac{\partial^2 q}{\partial x^2} \quad \text{\( \tau \)} = \frac{\partial^2 q}{2x^2} \quad \text{\( \tau \)} = \frac{\partial^2 q}{2x^2} \]

\[ \lim_{d \to 0} \frac{m}{d} \to 0 \]

\[ \therefore \text{ Eq. of Motion Becomes} \]

\[ \frac{\partial^2 q}{\partial t^2} = \frac{\partial^2 q}{\partial x^2} \quad \text{1D Wave Equation} \]

\[ \frac{\tau}{\tau} = \frac{v^2 s}{(velocity)^2} \]

\[ \frac{\partial^2 q}{\partial t^2} = \frac{v^2 \partial^2 q}{\partial x^2} \quad \text{Amplitude} \]

\[ \frac{\partial^2 q}{\partial t^2} = \frac{v^2 \partial^2 q}{\partial x^2} \quad \text{Spatial} \]

\[ \frac{v^2}{s} \quad \text{Velocity} \]
WAVE EQUATION SOLUTION (SOLVING FOR AMPLITUDE FN. \( q(x,t) \))

\[ q(x,t) = u(x) \cdot T(t) \]

\[ \frac{\partial^2 q}{\partial t^2} = v^2 \frac{\partial^2 q}{\partial x^2} \]

\[ u(x) \cdot \frac{\partial^2 T}{\partial t^2} = v^2 \cdot T(t) \cdot \frac{\partial^2 u(x)}{\partial x^2} \]

\[ \frac{1}{T(t)} \cdot \frac{\partial^2 T}{\partial t^2} = \frac{v^2}{u(x)} \cdot \frac{\partial^2 u(x)}{\partial x^2} = \text{Constant} = K \]

\[ \text{Independent of } x \]

\[ \text{Independent of } t \]

\[ \frac{\partial^2 T(t)}{\partial t^2} - v^2 K \cdot T(t) = 0 \]

\[ \frac{\partial^2 u(x)}{\partial x^2} - u(x) \cdot K = 0 \]

3 CASES FOR \( K \)

1) \( K = 0 \) \( \Rightarrow \) \( T(t) = a + b \) \( , \) \( u(x) = cx + d \)

\( c = d = 0 \) (FROM BOUNDARY CONDITIONS)
2) \( k > 0 \) \hspace{1em} \text{let} \hspace{1em} k = k^2, k \in \mathbb{R}

\[
\frac{\partial^2 u(x)}{\partial x^2} - k^2 u(x) = 0
\]

\( u(x) = u_0 e^{-\alpha x} \)
\( \alpha = \pm k \)

\( u(x) = u_0 e^{kx} + u_1 e^{-kx} \)

\text{using b.c.}

\( u(0) = u_0 e^0 + u_1 e^0 = u_0 + u_1 = 0 \)

\( u(k) = u_0 e^{k^2} + u_1 e^{-k^2} = 0 \)

\text{solving for} \hspace{1em} u_0 = u_1 = 0

\text{TENIAL! HOO44A}

3) \( k < 0 \) \hspace{1em} k = -k^2, k \geq 0, k \in \mathbb{R}

\[
\frac{\partial^2 u(x)}{\partial x^2} + k^2 u(x) = 0 \Rightarrow x = \pm ik, i = \sqrt{-1}
\]

\( u(x) = u_1 e^{ikx} + u_2 e^{-ikx} \)

\( u(0) = 0 = u_1 + u_2 \)

\( u(k) = 0 = u_1 e^{ikk} + u_2 e^{-ikk} \)

\( = u_1 (e^{ikk} - e^{-ikk}) = 0 \)

\text{either} \hspace{1em} u_1 = 0 \hspace{1em} \text{[TENIAL]} \hspace{1em} \text{or} \hspace{1em} e^{ikk} - e^{-ikk} = 0 \)
\[ e^{ikl} - e^{-ikl} = \sin kl = 0 \]

\[ kl = n\pi, \quad n \in \mathbb{Z}, \quad \forall \{k, l\} \]

\[ U(x) = U_1 \sin \left(\frac{n\pi}{l} x\right) \quad \text{if you couldn't tell, this is a sine wave.} \]

For time:

\[ T(t) = T_1 \sin (w_1 t) + T_2 \cos (w_2 t) \]

\[ w_n = \sqrt{k_n} = \frac{\sqrt{n\pi}}{l} \]

**Full Solution**

\[ q(x, t) = U(x) \cdot T(t) = \left[U_1 \sin \left(\frac{n\pi}{l} x\right)\right]\left[T_1 \sin (w_1 t) + T_2 \cos (w_2 t)\right] = \left[F \cos (w_1 t) + G \sin (w_2 t)\right] \sin \left(\frac{n\pi}{l} x\right) \]

Or for the nth component. Pulse comes from the fact that summing a sine + cosine wave results in another wave with some phase shift.

\[ q_n(x, t) = [A_n \cos (w_1 t + \phi_n)] \sin \left(\frac{n\pi}{l} x\right) \]

Any linear combination of \(q_n\)’s is also a solution.

\[ U(x, t) = \sum_{n=1}^{\infty} A_n \cos (w_1 t + \phi_n) \sin \left(\frac{n\pi}{l} x\right) \]

\[ \text{Intensity} \quad \text{Interference} \]
Each $q_n$ is called a normal mode B/C they correspond to a harmonic frequency

$$\frac{\omega_n}{2m} = \nu_n, \quad \frac{\nu_n}{2l}$$

$\begin{array}{l|ll}
 n = 1 & \text{fundamental} \\
 n = 2 & 2^{nd} \text{harmonic} \\
 & 1^{st} \text{overtone}
\end{array}$

---

(Place where no movement occurs)

$\begin{array}{l}
 n = 1 \\
 n = 2 \\
 n = 3
\end{array}$

$\begin{array}{l}
 \text{n-1 nodes} \\
 \text{node}
\end{array}$

$q(x,t)$ $\rightarrow$ wave amplitude

$|q(x,t)|^2$ $\rightarrow$ wave intensity

Although this wave equation has only real solutions, we will see wave functions with complex properties.

$$|q(x,t)|^2 = q^*(x,t) \cdot q(x,t) \quad (a+bi)^* = a - bi$$

$$|q^* \cdot q|^2 = (a+bi)^* (a+bi)$$

$$= (a - bi)(a + bi)$$

$$= a^2 + b^2$$
**Schrödinger Equation**

\[ \psi(x, t) = \psi(x) \cos \omega t \]  

Putting into wave equation

\[ \frac{\partial^2 \psi(x)}{\partial x^2} + \frac{\omega^2}{v^2} \psi(x) = 0 \]

\[ \omega = \frac{2 \pi v}{\lambda} \]

\[ \lambda = \frac{h}{p} \]

Find \( p \) from total energy

\[ E = \frac{1}{2} \frac{p^2}{m} + V(x) \]

\[ p = \sqrt{2m(E-V)} \]

\[ \lambda = \frac{h}{p} = \frac{h}{\sqrt{2m(E-V)}} \]

\[ \frac{\partial^2 \psi(x)}{\partial x^2} + \frac{4\pi^2}{\hbar^2} \left[ \frac{2m(E-V)}{\hbar^2} \right] \psi(x) = 0 \]