Commutator Formulas

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1 Introduction

A commutator is defined as

\[ [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \] (1)

where \( \hat{A} \) and \( \hat{B} \) are operators and the entire thing is implicitly acting on some arbitrary function. We’ve seen these here and there since the course began, most recently in the Heisenberg equation of motion for an expectation value’s time dependence:2

\[ \frac{d}{dt} \langle A \rangle_t = \frac{1}{i\hbar} \langle [\hat{A}, \hat{H}] \rangle_t \]

It’s worth recalling that any operator \( \hat{A} \) for which \( [\hat{A}, \hat{H}] = 0 \) is a constant of the motion – that is, its expectation value \( \langle A \rangle_t = \langle A \rangle \) will be independent of time altogether.

Commutators can be a little tricky (Problem Set 2 should’ve proved that to you!), so let’s have a look at the different formulas we can use to make manipulating them a little easier.

1 Most quantum mechanics books will discuss commutators in some detail. You can also check out the Wikipedia page, http://en.wikipedia.org/wiki/Commutator, for more information.

2 The notation I’m using here is shorthand. Wherever the \( \langle \ldots \rangle_t \) appears, you should remember that it means we’re calculating

\[ \langle \Psi(t)|\ldots|\Psi(t) \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)(\ldots)\Psi(x,t) \, dx \]
2 Formulas

There aren’t actually that many formulas to point out to you – “just” seven, not counting one that I’ll have you prove as an exercise – but the amazing thing is that only the definition given in (1) is fundamental! As a result, you should be able to prove any claim that you find dubious on your own, and you’ll only have to do a little pencil pushing. Even if you don’t have trouble believing any of these formulas, it’s worth working them out at least once.

With that said, here they are (c is a constant):

\[
\begin{align*}
[\hat{A}, c] &= 0 \\
[\hat{A}, \hat{A}] &= 0 \\
[\hat{A}, \hat{B}] &= -[\hat{B}, \hat{A}] \\
[c\hat{A}, \hat{B}] &= [\hat{A}, c\hat{B}] = c[\hat{A}, \hat{B}] \\
[\hat{A}, \hat{B} \pm \hat{C}] &= [\hat{A}, \hat{B}] \pm [\hat{A}, \hat{C}] \\
[\hat{A}\hat{B}, \hat{C}] &= \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B} \\
[\hat{A}, \hat{B}\hat{C}] &= \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}
\end{align*}
\]

3 Exercise

Show that \([\hat{A}, f(\hat{A})] = 0\), given that the function \(f\) has a power series expansion in \(\hat{A}\). This is a handy thing to know when you want to evaluate the commutator of \(\hat{x}\) or \(\hat{p}\) with the Hamiltonian operator \(\hat{H}\).